THE USE OF L-MOMENTS TO FIT THE GENERALIZED LAMBDA DISTRIBUTION TO SAMPLE DATA

THESIS

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Presented to the Faculty of the Graduate School of Engineering
of the Air Force Institute of Technology
Air University
In Partial Fulfillment of the
Requirements for the Degree of
Master of Science in Operations Research

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March, 1994

Approved for public release; distribution unlimited

THESIS APPROVAL

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Thesis Title: The Use of L-Moments to Fit the Generalized Lambda Distribution to Sample Data

Defense Date: 18 February 1994

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Acknowledgements

There are several people who have been very helpful to me during the course of my studies. My thesis committee members, Dr. Edward Mykytka and LtCol. Paul Auclair, were invaluable. They helped me focus when I lacked insight and offered me guidance when I went astray.

I would also like to thank all my classmates of the Graduate Strategic and Tactical sciences Program, class of '94. Associating with people who have a sense of teamwork and a sense of humor has made this a memorable part of my life.

Finally, I would like to thank the light of my life, Babette, for her patience and understanding. Without her love and support, this would have been an impossible task.

Robert Bruce Mohan

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Abstract

The Generalized Lambda Distribution (GLD) is a four parameter, continuous function which is flexible enough to approximate many different probability density functions (PDF). Like any distribution, the GLD can be fitted to sample data, and can be used to generate similar samples for use in computer simulations and models. The strengths of the GLD lie in its abilities to approximate many well-known distributions, represent data when the underlying distribution is unknown, and simply and efficiently fit or generate random variates. The method of moments is presently the most widely-accepted technique for estimating this distribution from sample data. However, sample moments are sensitive to extreme observations and are subject to large sampling variability as the size of the sample data decreases.

L-moments are expectations of certain linear combinations of order statistics. They can be defined for any random variable whose mean exists. L-moments form the basis of a general theory that summarizes and describes probability distributions and observed data samples. They can be used to estimate parameters and quantiles of probability distributions as well. Their main advantage over conventional moments is that, being linear, they suffer less from the effects of sampling variability, and are theoretically more robust to outliers in the data and small sample sizes.

The process of estimating the parameters of a probability distribution by matching its L-moments to those corresponding to sample data is referred to as the method of L-moments. This method has been successfully applied to many well-known probability distributions. In this thesis, the method of L-moments was applied to estimation of the Generalized lambda distribution. A Monte Carlo experiment was performed to compare the method of L-moments to the conventional method of moments and a method which uses alternate measures of symmetry and tailweight.

Experiment results showed that the method of L-moments is better than the conventional and alternate methods of moments for fitting distributions to sample data. The

improvement was more noticeable as the skewness and kurtosis of the empirical distribution of sample data increased. There was no change in the advantage of L-moments as the sample size changed. The experiment proved that L-moments suffer less from the effects of sampling variability, and that they are more robust to outliers in the data, when used to fit the GLD to sample data.

THE USE OF L-MOMENTS TO FIT THE GENERALIZED LAMBDA DISTRIBUTION TO SAMPLE DATA

I. Introduction

1.1 Simulation and the Lambda Distribution

Industry, commerce, government, and society in general all face problems which continue to grow in size and complexity. There is an obvious need for timely, cost-effective tools for solving these problems. One such tool is the computer-aided simulation model. Because of the rapid growth in the power and abilities of small computers, computer simulation models of dynamic, complex, or unique situations are more practical and easier to implement than ever.

The problem with simulation is that many applications require probabilistic models of input variables based on real-world data. Data such as the expected lifetime of a machine part, the number of troops needed to secure an objective, or the radiation pattern from a nuclear explosion are needed by the simulation to get results that "mean something." However, such information is rarely known with certainty. Instead, a planner may have a bank of accumulated data from previous tests and experiences which he can use to probabilistically describe the range of possible values for each of his variables of interest (Bergevin, 1993: 2).

The probability density function (PDF) summarily describes a continuous random variable. Many different PDF's, including those of well-known distributions as the Normal, Exponential, Gamma, and Weibull, are available to the modeler. Each PDF has its own distinct shape, or family of shapes, and covers a specific range of values. Since most commercial simulation packages provide a collection of PDF's, which together cover the ga[illegible] of these

well-known random variables, the modeler must decide which of the available distributions best "fits," or describes, his set of data.

An alternative to this collection of specialized PDF's is a distribution with a generalized density function that is not limited to a particular shape, but can instead take on a variety of shapes. Using such a generalized function frees the modeler from choosing an. competing distributions and allows him to possibly create an even better fit to the data.

The Generalized Lambda Distribution (GLD) has such a density function. The GLD is a four parameter, continuous distribution whose PDF is flexible enough to approximate many different probability density functions. The four parameters, λ_1, λ_2, λ_3, and λ_4, together define the location, scale, and shape of the density function. Like any distribution, the GLD can be fitted to sample data and used to generate similar samples for in computer simulations. The strengths of the GLD are its ability to approximate many well-known distributions, represent data when the underlying distribution is unknown, and simply and efficiently fit or generate random variates (Ramberg et al., 1979: 201).

The method of moments is the most widely-accepted technique for estimating the parameters of this distribution from sample data. Simply put, this method computes the first four sample moments: the mean, variance, skewness, and kurtosis. Then, a computerized search routine finds the GLD parameters that equate to those moments. Unfortunately, the higher-order sample moments, the skewness and kurtosis, are third- and fourth-power functions, respectively, and can vary greatly from sample to sample, especially when the sample size is small (Ramberg et al., 1979: 205). Recent research suggests that using Linear moments (L-moments) in place of conventional moments could eliminate the instability of the GLD brought on by the inherent variability of the higher-order c entional moments (Bergevin, 1993: 4).

L-moments are expectations of linear combinations of order statistics. They can be defined for any random variable whose mean exists. L-moments form the basis of a theory that summarizes and describes probability distributions and observed data samples. They

can be used to estimate parameters and quantiles of probability distributions as well. Their main advantage over conventional moments is that, being linear, they suffer less from the effects of sampling variability, and are theoretically more robust to outliers in the data (Hosking, 1990: 105).

Estimating the parameters of a distribution by matching its L-moments to those corresponding to sample data is referred to as the method of L-moments. This method has been successfully applied to many well-known probability distributions, such as the Normal, Gamma, Uniform, Exponential, Laplace, Logistic, Bernoulli, Wakeby and Weibull Distributions (Hosking, 1992: 186).

1.2 Applying Linear Moments

In theory, the method of L-moments could provide accurate estimates of the four lambda parameters of otherwise troublesome data samples. With accurate parameter estimates, the GLD should produce probability distribution functions for data samples from a wide variety of well-known distributions. In addition, the generated probability density function (PDF) should score well on accepted goodness-of-fit tests. The end result would be a GLD that produces an acceptable fitted PDF given any data sample, including small samples which previously produced poor estimates.

To date, researchers have proven the usefulness of the GLD in simulating other well-known distributions (Bergevin, 1993: 3). The GLD has been proven reliable when estimating distributions from large data samples using the method of moments. Other researchers have also proven the usefulness of the method of L-moments in estimating the parameters of well-known distributions from small data samples (Hosking, 1990: 112). However, no one has used the method of L-moments to estimate the parameters of the generalized lambda distribution, particularly with small data samples, although Hosking (1986) provides some initial work in this direction.

1.3 Overview

The purpose of thesis is to assess the usefulness of the method of L-moments applied to the GLD, especially in relation to other, more well-established methods for parameter estimation. One specific goal is to determine whether any one method is superior in fitting the generalized lambda distribution to sample data when the underlying distribution is unknown and the sample size is relatively small.

Chapter Two provides the background supporting this thesis. It presents the history of the GLD, methods of estimating the GLD from sample data, and an introduction to linear moments.

Chapter Three reviews a variation of the Lambda Distribution proposed by Hosking. It examines and reconciles the differences between Hosking's five-parameter Lambda Distribution and the four-parameter Generalized Lambda Distribution of Ramberg, Schmeiser, and Tukey. A resolution of the differences in the L-moment equations used for the two different distributions is presented. Finally, it compares the estimation techniques of linear moments developed by Hosking to two suggested by Mykytka and Bergevin.

Chapter Four compares the search routines used to solve the simultaneous equations which determine the lambda parameters from the sample L-moments. Hosking uses a Newton-Raphson iteration to find his five lambda parameters, while Mykytka and Ramberg use an adaptation of Powell's algorithm to find the four parameters of the GLD. The comparison examines it the accuracy of the algorithms, the computational speed of the routines, and the robustness of the functions, for each of the two methods.

Chapter Five discusses the design of a Monte Carlo experiment used to compare the effects of sampling variability on the method of linear moments, the method of conventional moments, and the method of alternate moments. The effectiveness of the three methods was evaluated based on visual comparisons of PDF's and Kolmogorov–Smirnov goodness-of-fit statistics. The experiment compared the three methods over three different distribution

shapes: Normal, Gamma, and Exponential; and three different sample sizes: $n = 25, n = 50$, and $n = 100$.

Chapter Six compares test results of the method of L-moments with those of the conventional method of moments and an alternate method proposed by Mykytka. It compares the goodness-of-fit using both Kolmogorov-Smirnov test statistics and visual plots. It also examines the method's stability with respect to small data samples by experimenting with sample sizes of 100, 50, and 25 elements. Finally, it examines the method's flexibility with respect to different distribution shapes by testing samples from the GLD's approximations to the Normal, Gamma, and Exponential distributions.

Chapter Seven discusses the conclusions drawn from the thesis, and makes recommendations for further research studies.

II. Discussion of the Literature

2.1 Chronological Organization

This chapter chronologically reviews major research milestones for both the Generalized Lambda Distribution and Linear Moments. It begins with the history and recent development of the lambda distribution. Although the L-moments were developed at the same time as the GLD, they were only recently applied to the generalized lambda distribution. It is from this point of discovery that the discussion of L-moments begins.

2.2 History of the Lambda Distribution

Although most continuous probability distributions are defined in terms of their density functions, f(x), or cumulative distribution functions, F(x), it is valid to define a distribution by its percentile function, if that percentile function exists. The percentile function is simply the inverse of the distribution function, *i.e.*, $R(p) = F^{-1}(p)$, *or equivalently*, $p = F(x)$. The percentile function determines the value, $x = R(p)$, such that the probability that a random variable having this distribution takes on a value less than x is p (Mykytka, 1979: 362). This relationship is depicted in Figure 2.1.

The ability to express a random variable in terms of its percentile function is quite useful in Monte Carlo simulation studies. In particular, it is well known that if R is the percentile function of a continuous probability distribution and if the random variable U is uniformly distributed on the (0,1) interval, then the transformation $X = R(U)$ yields a continuous random variable with percentile function R. Thus, since sources of uniform (0,1) pseudo-random variates are commonly available, this transformation yields a simple method for generating pseudo-random variates from distributions whose percentile functions are known and are computationally tractable (Ramberg et al., 1979: 202).

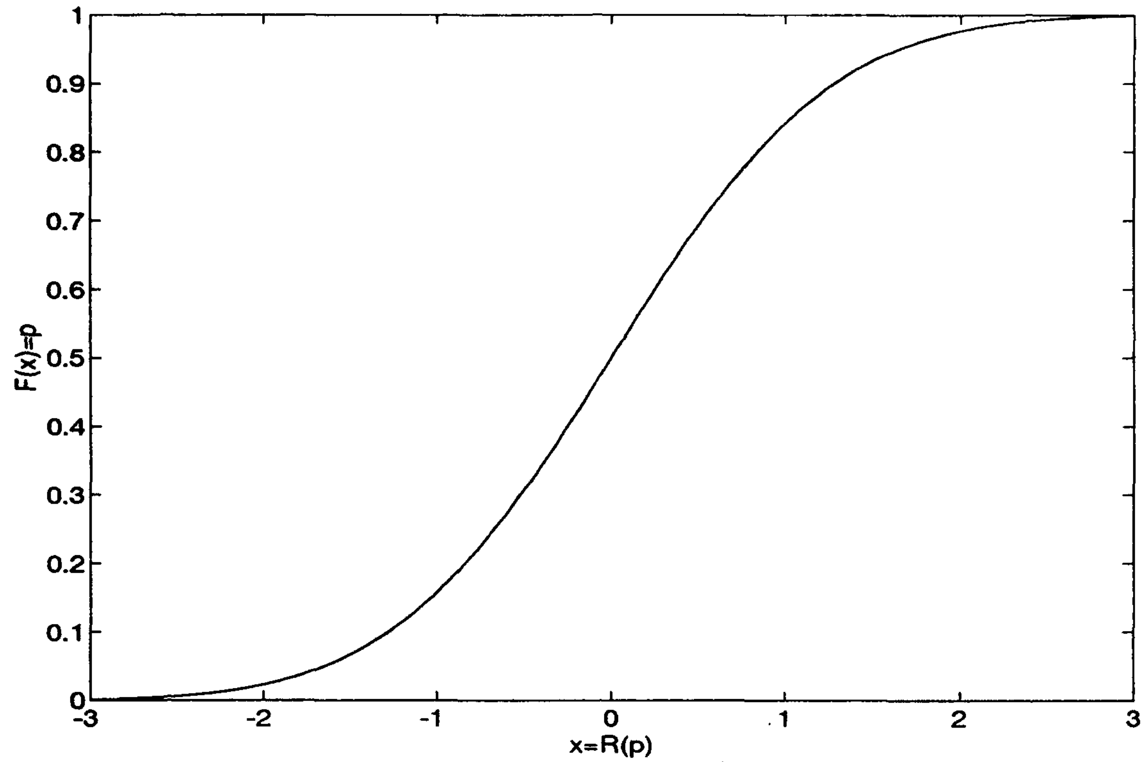

Figure 2.1 The relationship between the density function and the percentile function

Tukey (1960) created a function, which he called the lambda function, with this form. Tukey's function, which is valid for all non-zero λ, can be written as

$$R(p) = \frac{p^\lambda - (1-p)^\lambda}{\lambda}, \quad 0 \leq p \leq 1. \tag{2.1}$$

In 1976, Ramberg et al. (1979), generalized Tukey's single-parameter Lambda distribution and named it the Generalized Lambda Distribution. They recognized its strengths, by stating that "important applications of the distribution include the modeling and subsequent generation of random variates for simulation studies and Monte Carlo sampling studies of the robustness of statistical procedures" (Ramberg et al., 1979: 201). Mykytka (1978) further noted that "defined by its percentile function, the GLD provides a simple and

computationally efficient method for generating random variates having a wide variety of distributions."

Ramberg et al. generalized Tukey's function given by Equation 2.1 to a four-parameter form that could be used to approximate a number of well-known symmetric and asymmetric distributions. Their distribution is defined by the percentile function

$$R(p) = \lambda_1 + \frac{p^{\lambda_3} - (1-p)^{\lambda_4}}{\lambda_2}, \quad 0 \leq p \leq 1. \tag{2.2}$$

Ramberg et al. note that a close approximation to the standard normal distribution results when $\lambda_1 = 0$, $\lambda_2 = 0.1975$, and $\lambda_3 = \lambda_4 = 0.1350$ (Ramberg et al., 1979: 203). The distribution defined by Equation 2.2 is referred to as the Generalized Lambda Distribution. The GLD has also been referred to as the Ramberg–Schmeiser–Tukey (RST) distribution. The parameters λ_1 and λ_2 are location and scale parameters, respectively, while λ_3 and λ_4 are shape parameters that jointly determine the skewness and kurtosis of the GLD. When $\lambda_3 = \lambda_4$, the resulting density is symmetric.

Using the fact that $F(x) = p$ and $x = R(p)$, we can find the density function corresponding to Equation 2.2 by noting

$$f(x) = \frac{dF(x)}{dx} = \frac{dp}{dR(p)} = \left(\frac{dR(p)}{dp}\right)^{-1},$$

which yields

$$f(x) = \left(\frac{dR(p)}{dp}\right)^{-1} = \frac{\lambda_2}{\lambda_3 p^{\lambda_3 - 1} + \lambda_4 (1-p)^{\lambda_4 - 1}} \quad 0 \leq p \leq 1. \tag{2.3}$$

The cumulative distribution function of the GLD does not, in general, exist in a simple closed form, but this is not a cause for concern since it is also true of the Normal distribution, whose percentiles are more difficult to compute. For the GLD, it is simple to obtain plots of the distribution function by plotting p on the y–axis versus R(p) on the x–axis. Similarly, a

plot of the density function is obtained by plotting f[R(p)] on the y-axis against R(p) on the x-axis, for p ranging from zero to one. FORTRAN programs that compute R(p) and f[R(p)] for specified lambda values are given in Mykytka (1978: 82-84) and in the appendices of this study.

Ramberg and Schmeiser developed the following expressions for the mean, variance, skewness, and kurtosis of the GLD:

$$\mu = \mu(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \lambda_1 + \frac{A}{\lambda_2}, \tag{2.4}$$

$$\sigma^2 = \sigma^2(\lambda_2, \lambda_3, \lambda_4) = \frac{B - A^2}{{\lambda_2}^2}, \tag{2.5}$$

$$\alpha_3 = \alpha^3(\lambda_3, \lambda_4) = sign(\lambda_2) \cdot \frac{C - 3AB + 2A^3}{(B - A^2)^{\frac{3}{2}}}, \tag{2.6}$$

$$\alpha_4 = \alpha^4(\lambda_3, \lambda_4) = \frac{D - 4AC + 6A^2B - 3A^4}{(B - A^2)^2}, \tag{2.7}$$

where

$$A = \frac{1}{1 + \lambda_3} - \frac{1}{1 + \lambda_4}, \tag{2.8}$$

$$B = \frac{1}{1 + 2\lambda_3} - 2\beta(1 + \lambda_3, 1 + \lambda_4) + \frac{1}{1 + 2\lambda_4}, \tag{2.9}$$

$$C = \frac{1}{1 + 3\lambda_3} - 3\beta(1 + 2\lambda_3, 1 + \lambda_4) + 3\beta(1 + \lambda_3, 1 + 2\lambda_4) - \frac{1}{1 + 3\lambda_4}, \tag{2.10}$$

$$\begin{aligned} D = {} & \frac{1}{1 + 4\lambda_3} - 4\beta(1 + 3\lambda_3, 1 + \lambda_4) + 6\beta(1 + 2\lambda_3, 1 + 2\lambda_4) \\ & -4\beta(1 + \lambda_3, 1 + 3\lambda_4) + \frac{1}{1 + 4\lambda_4}, \end{aligned} \tag{2.11}$$

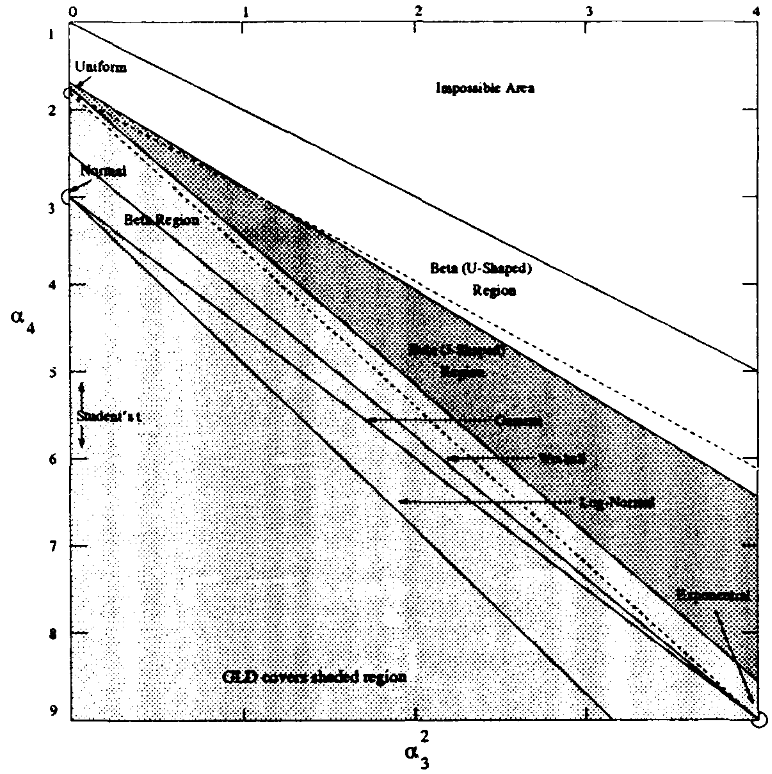

Figure 2.2 Characterization of Various Distributions by Skewness and Kurtosis

and $sign(\lambda_2)$ will be either 1 or -1, depending on whether the λ_2 parameter is positive or negative.

As the notation indicates, the skewness and kurtosis are functions of λ_3 and λ_4 alone. The skewness, α_3, is a function of only λ_3 and λ_4 since, as long as λ_3, $\lambda_4 > 0$, the sign of λ_2 is always the same as the sign of both λ_3 and λ_4 (Mykytka, 1978: 20). The variance, however, also depends on the shape parameter λ_2 and the mean depends on all four parameters. The shaded region of Figure 2.2 shows the different combinations of skewness and kurtosis that can be obtained from the GLD.

The technique known as the method of moments is the usual means of selecting the values of the GLD parameters. The method of moments begins by estimating the mean, variance, skewness, and kurtosis of a sample. Skewness and kurtosis are the standardized

third and fourth moments as defined by

$$\alpha_3 = \frac{E(x-\mu)^3}{\sigma^3}, \quad (2.12)$$

$$\alpha_4 = \frac{E(x-\mu)^4}{\sigma^4}. \quad (2.13)$$

Since the values for μ, σ^2, α_3, and α_4 are unknown, they must be estimated by the sample statistics $\overline{X}$, S'^2, $\hat{\alpha_3}$, and $\hat{\alpha_4}$, respectively. These sample moments are defined by

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad (2.14)$$

$$S'^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2, \quad (2.15)$$

$$\hat{\alpha_3} = \frac{\frac{1}{n}\sum_{i=1}^{n}(X_i-\overline{X})^3}{S'^3}, \quad (2.16)$$

$$\hat{\alpha_4} = \frac{\frac{1}{n}\sum_{i=1}^{n}(X_i-\overline{X})^4}{S'^4}. \quad (2.17)$$

Once the first four sample moments are computed, they are applied to Equations 2.4 through 2.11 to solve for the lambda parameters. Further, by choosing the four parameter values appropriately, a wide range of distributions can be duplicated, as indicated by the shaded portion of Figure 2.2

The biggest problem for the GLD, though, is the presence of variables raised to the third and fourth power, respectively, in the Equations for the third and fourth sample moments. Ramberg et al. said that "sample moments are sensitive to extreme observations, such as values more than two standard deviations from the mean, and sampling variability of the third and fourth moments can be large." (Ramberg et al., 1979: 205). This can, in turn, result in a poor, or even unacceptable, GLD fit. Mykytka recommends that the method of moments be used to fit data only when the sample size exceeds 150. (Mykytka, 1978: 60).

He also suggests that alternatives to the method of moments, like non-linear least squares or using four specified percentiles to solve for the parameters simultaneously, be explored (Mykytka, 1978: 80).

2.3 Estimating Distributions from Data

Mykytka and Ramberg acted upon their recommendations and experimented with the GLD using an alternative to the method of moments. They developed a new procedure using two statistics that are functions of order statistics and compared it to the method of moments (Mykytka, 1979: 361). The two statistics employed in the alternative procedure provide measures of the symmetry and tail weight of a distribution. Hence, they are analogous to the third and fourth moments, i.e., skewness and kurtosis. The two procedures were compared by a Monte Carlo study which examined the effects of sampling variability on them (Mykytka, 1979: 362).

For this effort, Mykytka and Ramberg chose two statistics developed by Hogg, Fisher, and Randles (1975) that are functions of linear combinations of the order statistics. To define these statistics, let $Y_1 < Y_2 < \ldots < Y_n$ be the order statistics of a random sample of size n from a continuous distribution. The skewness indicator, Q_3, is then defined by

$$Q_3 = \frac{\bar{U}(0.05)-\bar{M}(0.50)}{\bar{M}(0.50)-\bar{L}(0.05)}, \tag{2.18}$$

where $\bar{U}(0.05), \bar{M}(0.50)$, and $\bar{L}(0.05)$ are, respectively, the averages of the largest (uppermost) five percent of the $Y's$, the middle fifty percent of the $Y's$, and the smallest (lowest) five percent of the $Y's$. Similarly, Q_4, the tailweight descriptor, is defined as

$$Q_4 = \frac{\bar{U}(0.05)-\bar{L}(0.05)}{\bar{U}(0.50)-\bar{L}(0.50)}. \tag{2.19}$$

These, combined the first two conventional moments, the sample mean and variance, comprised the alternative method of estimation (Mykytka and Ramberg, 1979: 364). Mykytka's alternative method proved more accurate at estimating the GLD parameters of sample data than the conventional method of moments. From their Monte Carlo study they concluded that "the curve shapes of the densities estimated by the method of moments are more varied than the shapes of the densities estimated by the [alternate] method." This reduced variation in estimated densities suggests that sampling variability affects the alternate method less than it affects the method of moments. This alternative effectively expanded the range of allowable sample sizes because the alternative method more consistently estimated parameters for samples as small as fifty elements, as opposed to the method of moments, which was inconsistent even for samples of 100 elements (Mykytka, 1979: 366).

Ozturk and Dale (1985) developed and investigated the method of nonlinear least squares estimation, based on the suggestion of Ramberg et al. (1979). They used least squares estimation procedures to estimate the parameters of the GLD by minimizing the squared distance between the theoretical and empirical percentile functions. Although the least squares procedure offered no major computational advantage over the method of moments, it was easier to compute the objective function and the first and second lambda parameters (Ozturk and Dale, 1985: 81-82). Ozturk and Dale said that, "to compare the accuracy and precision of the fits obtained by the different methods, we performed a Monte Carlo simulation study similar to that made by Mykytka and Ramberg (1979). The results showed no appreciable differences, as long as the parameters could be estimated." (Ozturk and Dale, 1985: 83). The method of moments occasionally produces sample values of skewness and kurtosis for which the corresponding λ_3 and λ_4 parameters do not exist. Ozturk and Dale (1985) concluded that the least squares estimation procedure does not suffer from this problem.

Cheng (1985) developed an alternate procedure using the sample median and interquartile range to select the parameter values. She compared it to the method of moments by means of a Monte Carlo simulation study. Cheng concluded that the alternate procedure

appeared to be no better, and in some cases, slightly worse, than the method of moments. She also extended the concept to a procedure to select lambda parameter values which match specified percentiles of the distribution, but concluded that it was no more promising than her first procedure. Cheng used the chi–square goodness–of–fit test to compare the different methods, but had difficulty determining the actual degrees of freedom for the tests. She hypothesized that the problem was caused by a "large" number of parameters in the GLD. In this case, the test statistic may not possess an approximate chi–square distribution, or the method of estimating the number of degrees of freedom is inappropriate for this distribution. The fact that neither the method of moments nor the alternate method produce maximum-likelihood estimators also caused difficulty with the chi–square test (Cheng, 1985: 52-54).

Chou (1988) developed the method of maximum likelihood for determining the four lambda parameters of the GLD. She compared her method to the method of moments to determine which method produces better fits and better parameter estimates. Chou conducted a Monte Carlo experiment to perform the comparison. Her experimental results showed that the method of maximum likelihood appears to produce slightly better fits and better parameter estimates than the method of moments. Unfortunately, an in-depth study was restricted by the computational demands of the MLE method. Chou reports that the computer CPU time used by the MLE method is about one hundred times more than that used by the method of moments, and the CPU time needed grows exponentially with the number of observations in the sample. She, therefore, recommended the continued use of the method of moments (Chou, 1988: 70-71).

2.4 Computer Software for the Lambda Distribution

By 1991, the computer revolution had created an environment where work requiring a mainframe computer in 1979 could be done on a home computer. The state of the art in computer languages had progressed from FORTRAN 77 to object-oriented languages like C++.

Chung-Lung Hsu wrote a computer program in C++ in 1991, as a Master's thesis effort, which implemented methods for finding values for the lambda parameters. The methods computed parameters either from sample data or from various statistics, and used the lambda parameters to plot the curve of the probability distribution (Hsu, 1991: 2). When given sample data or statistics of the data, the program used these techniques to compute the lambda parameters: the method of moments; the method of moments with boundary points known; the method of percentiles; the replacement of mean and variance with the median and interquartile range; the replacement of skewness and kurtosis with alternate measures of symmetry and tailweight (Q_3, Q_4); and the method of least squares estimation (Hsu, 1991: 13-34).

All of these methods were suggested by Mykytka (Mykytka, 1978: 80-81) or Ramberg (Ramberg et al., 1979: 205). In fact, many of Hsu's C++ algorithms were adaptations of Mykytka's FORTRAN algorithms written in 1979. Hsu states that "all known methods of determining the lambda parameters have been introduced." (Hsu, 1991: 29). Despite the number of alternatives attempted, Hsu concluded that none was a significant improvement over the original method of moments first proposed by Ramberg (Hsu, 1991: 29-33).

2.5 Analysis of the Generalized Lambda Distribution

Bergevin (1993) analyzed the generalized lambda distribution intent on expanding the range of permissible skewness and kurtosis values used to compute lambda parameters. An increase in the parameter range increases the number and types of probability distributions that the GLD can approximate (Bergevin, 1993: 16). While he was not able to expand the coverage range of the GLD over all possible values of skewness and kurtosis, Bergevin successfully reduced the previously-uncovered range of values by approximately fifty percent (Bergevin, 1993: 53).

Because of its ability to "match moments," the GLD can mimic the behavior of most common PDF's by setting the four parameters to appropriate values. Figure 2.2 shows the ranges of these PDF's in terms of their measures of skewness and kurtosis. Some distri-

butions, such as the Uniform, Normal and Exponential, are represented by single points; others, such as the Student's t, Log-Normal and Gamma, are represented by curves; the three types of beta distributions are represented by regions of values. The top right-hand region, denoted the "Impossible Area," contains skewness-kurtosis combinations that will never be exhibited by any PDF. The GLD, at present, can "mimic" those PDF's found within the shaded regions of Figure 2.2. Obviously, all but some distributions with moments similar to those of the U-shaped beta distributions can currently be modeled using the GLD (Bergevin, 1993: 53).

The portion of his research that is critical to this study is the theory of L-moments. During an on-going literature search in the latter stages of his thesis effort, Bergevin uncovered publications of Hosking's current research of a new class of linear statistics, which Hosking named L-moments. While there was insufficient time to thoroughly develop an application of L-moments to the GLD, Bergevin conducted a promising initial analysis of the idea and derived the equations that express the theoretical L-moments as functions of the four lambda parameters (Bergevin, 1993: 59-60).

2.6 Derivation of the GLD from L-Moments

Hosking defines the first four L-moments via

$$\Lambda_1 = \int_0^1 R(p)dp, \tag{2.20}$$

$$\Lambda_2 = \int_0^1 R(p) \cdot (2p-1)dp, \tag{2.21}$$

$$\Lambda_3 = \int_0^1 R(p) \cdot (6p^2 - 6p + 1)dp, \tag{2.22}$$

$$\Lambda_4 = \int_0^1 R(p) \cdot (20p^3 - 30p^2 + 12p + 1)dp, \tag{2.23}$$

where R(p) is simply the percentile function.

As with the commonly-used measures of skewness and kurtosis, Hosking (1986: 7) chose to define the two higher order moments as dimensionless ratios relative to the second

order moment

$$\tau_3 = \frac{\Lambda_3}{\Lambda_2},$$

$$\tau_4 = \frac{\Lambda_4}{\Lambda_2}.$$

This research conforms to that convention.

The four L–moments have roles similar to those of conventional moments. Since the first L–moment is simply the expected value of the distribution, it is identical to the mean of the distribution. Also, a symmetric distribution will have $\tau_3 = 0$, just as $\alpha_3 = 0$ for symmetric distributions using conventional moments. According to Hosking, however, the estimators of τ_3 and τ_4 are more stable measures than $\hat{\alpha_3}$ and $\hat{\alpha_4}$, and are better estimates of symmetry and tailweight of the underlying theoretical PDF of a distribution.

For the GLD, the L-moment Equations become

$$\Lambda_1 = \lambda_1 + \frac{\lambda_4 - \lambda_3}{\lambda_2(\lambda_3+1)(\lambda_4+1)}, \tag{2.24}$$

$$\Lambda_2 = \frac{\lambda_3(\lambda_4+1)(\lambda_4+2)+\lambda_4(\lambda_3+1)(\lambda_3+2)}{\lambda_2(\lambda_3+1)(\lambda_3+2)(\lambda_4+1)(\lambda_4+2)}, \tag{2.25}$$

$$\tau_3 = \frac{(\lambda_3^2-\lambda_3)(\lambda_4+1)(\lambda_4+2)(\lambda_4+3)-(\lambda_4^2-\lambda_4)(\lambda_3+1)(\lambda_3+2)(\lambda_3+3)}{(\lambda_3+3)(\lambda_4+3)[\lambda_3(\lambda_4+1)(\lambda_4+2)+\lambda_4(\lambda_3+1)(\lambda_3+2)]}, \tag{2.26}$$

$$\tau_4 = \frac{(\lambda_3^3-3\lambda_3^2+2\lambda_3)(\lambda_4+1)(\lambda_4+2)(\lambda_4+3)(\lambda_4+4)+(\lambda_4^3-3\lambda_4^2+2\lambda_4)(\lambda_3+1)(\lambda_3+2)(\lambda_3+3)(\lambda_3+4)}{(\lambda_3+3)(\lambda_3+4)(\lambda_4+3)(\lambda_4+4)[\lambda_3(\lambda_4+1)(\lambda_4+2)+\lambda_4(\lambda_3+1)(\lambda_3+2)]}. \tag{2.27}$$

Bergevin's thesis contains a complete derivation of these equations (1993: appendix A). Although Equations 2.24 through 2.27 look complicated, they are all composed of polynomials. Unlike the GLD Equations (Equations 2.4 through 2.11) based on conventional moments, these have no beta functions. Since computerized solution algorithms for systems

of polynomial equations are fairly common and easily adaptable. it is easier to determine the values of the GLD parameters using L-moments instead of conventional moments.

2.7 (The Problem of) Estimating L-Moments

Bergevin's review of the available literature did not reveal how to estimate the L-moments from sample data. Thus, he hypothesized two ways of doing so.

First, he noted that Hosking provides an alternate definition of the L-moments based on the order statistics:

$$\Lambda_1 = E(X), \tag{2.28}$$

$$\Lambda_2 = \tfrac{1}{2}E(X_{2:2} - X_{1:2}), \tag{2.29}$$

$$\Lambda_3 = \tfrac{1}{3}E(X_{3:3} - 2X_{2:3} + X_{1:3}), \tag{2.30}$$

$$\Lambda_4 = \tfrac{1}{4}E(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4}), \tag{2.31}$$

where $X_{a:b}$ denotes the a^{th} order statistic in a sample of size b. Then he suggested that one could define estimators of these based on the average of the results of every possible subset of the proper size within the data set. For example, in calculating the value Λ_2 for a sample of size $n = 10$, one could estimate the value of Λ_2 by computing one–half of the difference between the largest and smallest observations within all $\binom{10}{2}$ possible subsets of size 2 selected from the 10 observations and then averaging the results.

His second suggestion was to derive the empirical CDF of the data set, and create finite summations to approximate Equations 2.20 through 2.23. This seemed to be the more reasonable approach, since the work required is less than that involved in calculating each possible subset of sizes 2, 3, and 4.

If an effective means of calculating sample L-moments can be found, then Equations 2.24 through 2.27 can replace the more complicated GLD moment equations to estimate appropriate values for the λ_i. Of course. the quality of these two possible estimation proce-

dures is yet unproven: no one knows whether either method produces unbiased estimators or whether they would provide a good fit to the data sample or population (Bergevin, 1993: 60-61). The primary objective of this thesis is to assess the latter.

2.8 Another Estimation Method

In his 1986 report which introduced the notion of L-moments, Hosking proves mathematically and by example that sample L-moments, being linear functions of the data, suffer less from the effects of sampling variability than do conventional sample moments. They are more robust to outliers in the data, and they enable more secure inferences to be made from small samples about an underlying probability distribution. He claims that L-moments frequently yield better parameter estimates than conventional moments (Hosking, 1986: 1). In some cases, L-moments can specify a distribution even if some of its conventional moments do not exist. In fact, all linear moments exist if and only if the mean exists (Hosking, 1986: 7). Hosking concluded "that the main features of a probability distribution should be well summarized by the following four measures: the mean or L-location, λ_1; the L-scale, λ_2; the L-skewness, τ_3; and the L-kurtosis, τ_4 (Hosking, 1986: 9).

Hosking further proposes a method of estimating L-moments from a sample, and of calculating the parameters of the GLD from those sample L-moments. These aspects of the method of L-moments are discussed at length in Chapter Three.

III. Hosking's Lambda Distribution and L-Moments

3.1 Comparing Ramberg's and Hosking's Lambda Functions

Hosking developed the L-moment equations for sixteen specific distributions, including a form of the Generalized Lambda Distribution. The five–parameter form of the GLD he chose is given by (Hosking, 1986: 83) as

$$x(F) = \xi + \alpha F^{\beta} - \gamma(1-F)^{\delta}, \quad 0 \leq F \leq 1. \tag{3.1}$$

Comparing Hosking's notation with that more commonly used by Ramberg et al. for the GLD, and assuming that $\alpha = \gamma$ as Hosking does, reveals that $F = p$, $x(F) = R(p)$, $\xi = \lambda_1$, $\alpha = \gamma = \frac{1}{\lambda_2}$, $\beta = \lambda_3$, $\delta = \lambda_4$. Substituting Ramberg's notation in Equation 3.1 results in

$$R(p) = \lambda_1 + \frac{p^{\lambda_3} - (1-p)^{\lambda_4}}{\lambda_2}, \quad 0 \leq p \leq 1, \tag{3.2}$$

the usual percentile function of the four-parameter GLD (Bergevin, 1993: 7). Hosking later altered his quantile function to ensure its continuity as β and δ approach zero. The altered function, shown in Equation 3.3, and its resulting L-moment equations are suitable for a Newton-Raphson search algorithm (Hosking, 1993: 1),

$$x(F) = \xi + \frac{\alpha \cdot F^{\beta-1}}{\beta} + \frac{\gamma \cdot (1-(1-F)^{\delta})}{\delta}, \quad 0 \leq F \leq 1. \tag{3.3}$$

This form of the Lambda Distribution is implemented in Hosking's Lambda Parameter Estimation FORTRAN subroutine found in Appendix N.

3.2 Comparing Bergevin's and Hosking's L-Moment Equations

Using the definitions of the first four L-moments given by Equations 2.20 through 2.23, Bergevin derived the equations listed below for the first four L-moments of the GLD

(Bergevin, 1993: 59):

$$\Lambda_1 = \lambda_1 + \frac{\lambda_4 - \lambda_3}{\lambda_2(\lambda_3+1)(\lambda_4+1)} \tag{3.4}$$

$$\Lambda_2 = \frac{\lambda_3(\lambda_4+1)(\lambda_4+2) + \lambda_4(\lambda_3+1)(\lambda_3+2)}{\lambda_2(\lambda_3+1)(\lambda_3+2)(\lambda_4+1)(\lambda_4+2)} \tag{3.5}$$

$$\Lambda_3 = \frac{(\lambda_3^2-\lambda_3)(\lambda_4+1)(\lambda_4+2)(\lambda_4+3)-(\lambda_4^2-\lambda_4)(\lambda_3+1)(\lambda_3+2)(\lambda_3+3)}{\lambda_2(\lambda_3+1)(\lambda_3+2)(\lambda_3+3)(\lambda_4+1)(\lambda_4+2)(\lambda_4+3)} \tag{3.6}$$

$$\Lambda_4 = \frac{(\lambda_3^3-3\lambda_3^2+2\lambda_3)(\lambda_4+1)(\lambda_4+2)(\lambda_4+3)(\lambda_4+4)+(\lambda_4^3-3\lambda_4^2+2\lambda_4)(\lambda_3+1)(\lambda_3+2)(\lambda_3+3)(\lambda_3+4)}{\lambda_2(\lambda_3+1)(\lambda_3+2)(\lambda_3+3)(\lambda_3+4)(\lambda_4+1)(\lambda_4+2)(\lambda_4+3)(\lambda_4+4)}. \tag{3.7}$$

The equations for the first four L-moments which Hosking derived from his five-parameter form of the GLD are (Hosking, 1986: 83)

$$\Lambda_1 = \xi + \frac{\alpha}{1+\beta} - \frac{\gamma}{1+\delta}, \tag{3.8}$$

$$\Lambda_2 = \frac{\alpha\beta}{(1+\beta)(2+\beta)} + \frac{\gamma\delta}{(1+\delta)(2+\delta)}, \tag{3.9}$$

$$\Lambda_3 = \frac{\alpha\beta(\beta-1)}{(1+\beta)(2+\beta)(3+\beta)} - \frac{\gamma\delta(\delta-1)}{(1+\delta)(2+\delta)(3+\delta)}, \tag{3.10}$$

$$\Lambda_4 = \frac{\alpha\beta(\beta-1)(\beta-2)}{(1+\beta)(2+\beta)(3+\beta)(4+\beta)} + \frac{\gamma\delta(\delta-1)(\delta-2)}{(1+\delta)(2+\delta)(3+\delta)(4+\delta)}. \tag{3.11}$$

There are apparent similarities between the L-moment equations for these forms of the GLD. Appendix A contains detailed proofs of the equivalence of each of the four pairs of equations.

3.3 Comparing Sample L-Moments

Nowhere in his various papers and reports on L-moments does Hosking thoroughly discuss estimating L-moments from sample data. In one of his most specific statements, he notes that the sample L-moments can be defined as U-statistics [a concept originally

introduced by Hoeffding (1948)] as follows:

$$L_1 = n^{-1} \sum_i x_i, \tag{3.12}$$

$$L_2 = \frac{1}{2} \binom{n}{2}^{-1} \sum_{i>} \sum_{j} (x_i - x_j), \tag{3.13}$$

$$L_3 = \frac{1}{3} \binom{n}{3}^{-1} \sum_{i>} \sum_{j>} \sum_{k} (x_i - 2x_j + x_k), \tag{3.14}$$

$$L_4 = \frac{1}{4} \binom{n}{4}^{-1} \sum_{i>} \sum_{j>} \sum_{k>} \sum_{l} (x_i - 3x_j + 3x_k - x_l). \tag{3.15}$$

The equation for the first sample L-moment is simply the equation for the sample mean. The second equation implicitly computes the average of the differences between the largest and smallest observations within all subsets of size 2 from the sample. The third and fourth equations similarly turn compute the average values of certain functions of the order statistics within all subsets of sizes 3 and 4, respectively. These four equations turn out to implement Bergevin's suggestion for estimating the L-moments based on such order statistics, as discussed in Chapter Two.

It is not clear, however, that this is the procedure actually implemented by Hosking since his FORTRAN L-moment subroutines (obtained from the Internet Statistics Library at Carnegie–Mellon University and listed in Appendix M) appear to have a very different structure. In fact, the first step in Hosking's algorithm computes what he calls a "plotting position" which is intended to compensate for the fact that the finite data sample comes from an infinite population and allows for distributions with infinite tails (Hosking, 1986: 16). This is followed by a number of manipulations that appear to be performed for the sake of computational efficiency. Unfortunately, these obscure the exact procedure being implemented. It does appear, however, that this approach is tantamount to the computation of the L-moments of the empirical CDF, as also suggested by Bergevin. Isolated comments made by Hosking suggest that these two approaches are equivalent.

IV. Lambda Parameter Search Routines

4.1 Hosking's Lambda Quantile Function

Recall that, as discussed in Chapter Three, Hosking revised the quantile function of the five–parameter version of the GLD to have the form shown in Equation 3.3. Prior to this revision, the denominators of the third and fourth L-moments, τ_3 and τ_4, were zero when either β or δ was zero, creating a discontinuity. In the revised form, since τ_3 and τ_4 are continuous functions of β and δ, the denominators of τ_3 and τ_4 equal one when β and δ equal zero. The equations Hosking uses for the third and fourth L-moments, τ_3 and τ_4, in his FORTRAN search routine (Appendix N) are

$$\tau_3 = \frac{-\left[\left(\frac{1}{1+\beta}\right)\left(\frac{1}{2+\beta}\right)\left(\frac{1}{3+\beta}\right)(1-\beta)\right]+\left[\left(\frac{1}{1+\delta}\right)\left(\frac{1}{2+\delta}\right)\left(\frac{1}{3+\delta}\right)(1-\delta)\right]}{\left[\left(\frac{1}{1+\beta}\right)\left(\frac{1}{2+\beta}\right)\right]+\left[\left(\frac{1}{1+\delta}\right)\left(\frac{1}{2+\delta}\right)\right]} \tag{4.1}$$

and

$$\tau_4 = \frac{-\left[\left(\frac{1}{1+\beta}\right)\left(\frac{1}{2+\beta}\right)\left(\frac{1}{3+\beta}\right)\left(\frac{1}{4+\beta}\right)(1-\beta)(2-\beta)\right]+\left[\left(\frac{1}{1+\delta}\right)\left(\frac{1}{2+\delta}\right)\left(\frac{1}{3+\delta}\right)\left(\frac{1}{4+\delta}\right)(1-\delta)(2-\delta)\right]}{\left[\left(\frac{1}{1+\beta}\right)\left(\frac{1}{2+\beta}\right)\right]+\left[\left(\frac{1}{1+\delta}\right)\left(\frac{1}{2+\delta}\right)\right]}. \tag{4.2}$$

Being a function of only two variables, β and δ, the equations for τ_3 and τ_4 lend themselves well to solution of two simultaneous equations for two unknowns. Given the first four sample L-moments, the lambda parameters can be calculated in progression, starting with the two equations above. Once β and δ have been determined, the equation for the second L-moment can be solved directly for α and γ assuming, as Hosking does, that $\alpha = \gamma$. Then, the equation for the first L-moment, Λ_1, can be solved explicitly for ξ.

4.2 Hosking's Solution of Simultaneous Equations

Hosking's algorithm in his FORTRAN subroutine, PELTU4 (Appendix N) employs a form of the Newton–Raphson method to implicitly solve Equations 4.1 and 4.2 simultaneously for β and δ. The routine finds an approximate solution to the nonlinear system

$F(\beta, \delta) = 0$, where

$$F(\beta, \delta) = (\frac{L_3}{L_2} - \tau_3) + (\frac{L_4}{L_2} - \tau_4), \tag{4.3}$$

and an initial $(\beta, \delta) = (0, 0)$. The expressions $\frac{L_3}{L_2}$ and $\frac{L_4}{L_2}$ are the known sample L-moments, while τ_3 and τ_4 are calculated from Equations 4.1 and 4.2

Hosking's algorithm iteratively searches for the values of β and δ that yield $F(\beta, \delta)$ within a pre-determined tolerance of zero. Hosking determines the stepwise changes in β and δ by solving the derivatives of the L-moment equations, finding the inverse of the matrix of derivatives, and incrementing the solution vector (β, δ) with the vector product of $\left[\frac{L_3}{L_2}, \frac{L_4}{L_2}\right]^T$ and the matrix inverse.

In his PELTU4 subroutine, Hosking chose to initiate the iterative search from the starting point $(\beta, \delta) = (0, 0)$ because both β and δ are constrained to the region $-1 < x < 1$. Hosking intentionally constrains the parameters to these values since, if β and δ are unique, they lie in this region. The choice to begin the search at the origin necessitated the use of the revised quantile function discussed in the previous section. Hosking allows the search to proceed through twenty iterations before terminating the algorithm and returning an error message that the algorithm failed to converge on an acceptable solution.

4.3 The Generalized Lambda Percentile Function

As mentioned earlier, Bergevin used Ramberg's percentile function of the four-parameter GLD, Equation 2.2, to derive the equations for Hosking's linear moments in terms of the four lambda parameters. He derived Equations 2.26 and 2.27 for the third and fourth L-moments, τ_3 and τ_4 (Bergevin, 1993: 7, 63). Like Hosking's equations for τ_3 and τ_4, the two equations are in terms of only two variables, λ_3 and λ_4. Thus, these equations can also be solved simultaneously for two unknowns. Given the first four sample L-moments, the lambda parameters can be calculated in progression: Once λ_3 and λ_4 have been calculated, the equation for the second L-moment can be solved directly for λ_2. The equation for the first L-moment, can then be solved explicitly for λ_1.

4.4 Mykytka's Solution of Simultaneous Equations

Mykytka and Ramberg (Mykytka and Ramberg, 1979) and Mykytka (Mykytka, 1978) used non-linear programming methods to find the minimum sum-of-squared errors between the calculated and desired values of the third and fourth conventional moments, α_3 and α_4

$$Min \quad f(\lambda_3, \lambda_4) = (\alpha_3(\lambda_3, \lambda_4) - \alpha_3)^2 + (\alpha_4(\lambda_3, \lambda_4) - \alpha_4)^2 \tag{4.4}$$

$$subject\ to \qquad \lambda_3 \cdot \lambda_4 \geq 0. \tag{4.5}$$

Expression 4.5 ensures that λ_3 and λ_4 have the same sign. The expression in Equation 4.4 can be minimized using Powell's Algorithm for non-linear function minimization. This algorithm was originally designed for unconstrained non-linear function minimization. However, the GLD is constrained by two requirements. First, λ_3 and λ_4 must have the same sign. Second, the objective is to minimize the sum-of-squared errors between the desired and calculated values of α_3 and α_4 subject to this constraint (see Equations 4.4 and 4.5). In the current implementation, "unacceptable" values—those combinations that do not satisfy Equation 4.5—are eliminated from consideration by replacing their sum-of-squared errors with a large penalty. This penalty is enforced by defining the objective function as

$$min\ Z = f(\lambda_3, \lambda_4)$$

where

$$\begin{aligned} Z &= [(\alpha_3(\lambda_3, \lambda_4) - \alpha_3)^2 + (\alpha_4(\lambda_3, \lambda_4) - \alpha_4)^2]; & \lambda_3 \cdot \lambda_4 \geq 0 \\ Z &= 10; & \lambda_3 \cdot \lambda_4 < 0. \end{aligned}$$

Since an appropriate "match" produces the desired α_3 and α_4 values only when the optimal value of the objective is zero (practically interpreted as having an objective function value less than 0.0002), one can avoid "unacceptable" combinations of λ_3 and λ_4 by using this penalty (Bergevin, 1993: 18).

When using the FORTRAN program given by Mykytka and Ramberg, the user must input an initial "guess" at the values for λ_3 and λ_4. Since λ_3 and λ_4 must have the same sign, their initial values must be either both positive or both negative. It can be shown that even if the initial guess is in the wrong region the algorithm will sometimes switch regions to find the optimal solution. For example, an initial guess with λ_3 and λ_4 both negative could be provided when the optimal solution actually has them both positive. But because of the algorithm's search technique, the possibility of jumping directly from one acceptable region to the other is slight. That is, although the algorithm sometimes does switch regions, it usually does not. If the user does not achieve a satisfactory Z–value when starting in one region, he need simply apply the algorithm a second time, changing his starting point to one in the opposite region to produce a better objective function value (Bergevin, 1993: 19). After discovering this, Mykytka revised the FORTRAN algorithm to first search from an initial point in one region, ($\lambda_3 = 0.05, \lambda_4 = 0.05$). If the first search fails to converge to the solution, the algorithm begins another search from another region, ($\lambda_3 = -0.05, \lambda_4 = -0.05$) to attempt to reduce the tolerance of the solution.

A major part of Bergevin's thesis was to study and test Powell's Algorithm for its effectiveness and robustness when searching for lambda parameters, especially near the boundaries of the feasible regions. Bergevin showed that Powell's algorithm does not significantly limit the GLD. Even in cases where the final λ_3 and λ_4 values lie on (or near) the limiting boundaries, the algorithm was not affected by the penalty function. The algorithm also performed just as well in an unconstrained search as it did in the constrained case (Bergevin, 1993: 32). Mykytka's FORTRAN subroutines (Appendices O and P) compute the four GLD parameters from the first four conventional moments of sample data. Because L-moments so closely parallel the conventional moments, it was a simple matter to adapt the subroutines to the L-moment application. The only changes needed were to replace the conventional moment equations with the simpler L-moment equations in the CALCFX subroutine and the FN2 subroutine (Appendix O). The primary code for Powell's algorithm in the BOTM subroutine (Appendix P) was unchanged.

4.5 Comparing Newton's Method and Powell's Algorithm

In comparing Hosking's Newton–Raphson algorithm to Powell's algorithm, three areas were examined. Most important of these was the robustness of the algorithm when searching for lambda parameters near the limits of the feasible regions. Second was the accuracy of the lambda parameters computed for the sample L-moments. The final area was the speed and efficiency of the computer search subroutines.

4.5.1 Robustness. The most important criteria for a search method is its ability to converge to a solution. A search method will most likely encounter difficulty when it approaches the boundary of the feasible region or a point of discontinuity of a function. Both Newton's method and Powell's algorithm must deal with boundary conditions as the parameters approach 1 or -1. Bergevin's equations also contain a discontinuity when the lambda parameters approach zero.

While the Newton–Raphson algorithm is a "classic" numerical method, Powell's algorithm benefits from Mykytka's and Bergevin's extensive verification and validation efforts for this particular application. Powell's algorithm is better for dealing with discontinuous and multi-modal objective functions (Bergevin, 1993: 29), and is appropriate to Bergevin's discontinuous L-moment equations. In addition, during comparison tests of the two methods, the Newton–Raphson method sometimes converged on an unacceptable solution ($\lambda_3 > 0, \lambda_4 < 0$) for Gamma distributions. Newton's method also failed to converge on a solution for some Beta and Exponential distributions. Both of these distributions have parameter values which lie near the region boundaries. For these reasons, Powell's algorithm is considered to be more robust than the Newton–Raphson method.

4.5.2 Accuracy. To test the accuracy of each algorithm, a FORTRAN subroutine was developed to re-compute the L-moments, given the estimated lambda parameters (Appendix Q). When comparing the re-computed L-moments to the original sample L-moments, Powell's algorithm was consistently accurate to six decimal places on all four L-moments.

Newton's method was also accurate to six decimal places when computing the first and second L-moments. But, it was accurate to only four decimal places, and occasionally accurate to only two decimal places on the third and fourth L-moments. But when the lambda parameters from both routines were used to plot the PDF's of their respective distributions, there was no visible difference between the plots. Therefore, the results provided by the Newton–Raphson method and Powell's algorithm were equally accurate.

4.5.3 Speed and Efficiency. The Newton–Raphson method benefits from simplicity and quadratic convergence. In tests, Hosking's routine typically converged to an answer in less than ten iterations, for all types of distributions. This fast rate of convergence, combined with a small number of instructions, makes the Newton–Raphson method quite fast and efficient. Powell's algorithm was technically less efficient, sometimes requiring sixty iterations to converge to a solution. This, coupled with about three times as many instructions per iteration as the Newton–Raphson routine, makes Powell's algorithm slower and less efficient. Still, the overall run time for either routine is a matter of microseconds of CPU time.

Both routines are part of a program which runs in real-time and interacts with a human operator and batch files stored in memory. In this type of application, both input/output to memory and wait times from screen prompt to keyboard input are long compared to actual CPU time spent executing instructions. During testing, there was no discernible delay between the last keyboard input and the screen prompt for the next input, using either algorithm. So, while the Newton–Raphson method is technically faster than Powell's algorithm, they are practically both instantaneous. Therefore, there is no discrimination between the two based on speed and efficiency.

In summary, Powell's algorithm is more robust than the Newton–Raphson method. Both methods are equally suitable with respect to accuracy, speed, and efficiency. But, since robustness is the most important quality, Powell's algorithm is the overall best choice for a computer search routine.

V. Design of the Monte Carlo Experiment

5.1 Monte Carlo Experiments

A Monte Carlo simulation study was done to compare the effects of sampling variability in the method of linear moments, the conventional method of moments, and the alternative method of moments using Q_3 and Q_4 as measures of symmetry and tailweight. Henceforth, these methods are referred to as linear moments, conventional moments, and alternate moments, respectively. Each of the three methods was used to estimate the lambda parameters from nine different sets of thirty random samples each. The nine sets were combinations of three different Lambda distributions approximating the Normal, Gamma, and Exponential distributions, respectively; and three different sample sizes: $n = 25, n = 50$, and $n = 100$. Random samples were generated using the GLD quantile function to transform uniform [0,1] pseudo–random variates into variates from the selected distribution. The pseudo–random number generator DURAND, developed by IBM, was employed and is described in Appendix K. The same thirty seed values were used to generate the thirty samples for each of the nine sample sets (the seed values used were 1010, 1020, 1030, ... , 1300).

The sampling variability effects were compared by plotting – for each method, distribution, and sample size – the estimated probability density functions (PDF) for the first twenty–five of the thirty replications on the same graph. The software package used for plotting the PDF's limited the number of simultaneous plots to twenty–five. Kolmogorov–Smirnov goodness–of–fit statistics were also collected from each replication by comparing the fitted cumulative distribution functions (CDF) to that of the underlying theoretical distribution from which the sample was randomly generated.

The overall goal of this study was to determine whether or not the method of linear moments is less susceptible to sampling variability effects than the method of conventional moments. Therefore, the results of the Monte Carlo experiments using conventional moments were used as a "control group," and the results of the other experiments were compared to

these baseline experiments. Thus, the conclusions drawn were expressed in relative, rather than absolute measures.

5.2 Data Sample Sizes

Because the goal is to improve the GLD's ability to estimate the parameters of the underlying distributions from small samples and reduce the sampling variability effect, it is necessary to experiment with "small" sample sizes. Mykytka (1978:60) had previously determined that conventional moments were adequate for sample sizes as small as 150, and that smaller sample sizes introduced unacceptable sampling variability. Therefore, sample sizes of $n = 25, n = 50$, and $n = 100$ were selected as representative "small" sample sizes. These sample sizes are also more in line with samples being used in today's modeling and testing communities because of cost and time constraints.

5.3 Distributions Selected

In order to get a representative cross–section of various distribution types and shapes, three Lambda distributions whose PDF's approximate the shapes of Normal, Gamma, and Exponential PDF's, respectively, were chosen. These three distributions are commonly–used and well–known to anyone familiar with statistics. As seen in Figure 2.2, they represent a broad range of possible combinations of skewness and kurtosis. These three Lambda distributions were chosen to have the same first four moments as their respective Normal, Gamma, and Exponential counterparts. To remind the reader that they are the Lambda approximations to these distributions, they will be referred to with their names in quotes, i.e., as the "Normal," "Gamma," and "Exponential" distributions, respectively. Table 5.1 summarizes the characteristics of the chosen distributions.

Parameter	Approximate Distribution Shape		
	"Normal"	"Gamma"	"Exponential"
Mean	0	0.8	1
Variance	1	1	1
Skewness	0	$\sqrt{2}$	2
Kurtosis	3	6	9
λ_1	0	0	0
λ_2	0.1975	0.04134	-0.001632
λ_3	0.1349	0.005674	-0.9159×10^-5
λ_4	0.1349	0.04046	-0.001621
Λ_1	0	7.6558×10^-5	6.0309×10^-5
Λ_2	0.56382	0.52905	0.50065
τ_3	0	0.23425	0.33031
τ_4	0.12447	0.15455	0.16723
μ	0	0.8	1
σ^2	1	1	1
Q_3	1	2.6481	4.4743
Q_4	2.5959	2.7856	2.8676

Table 5.1 Moments and parameters of distributions used in the experiment

5.4 Methods of Fit

The purpose of this study was to determine whether or not the method of linear moments is a better method for fitting a distribution to sample data than the method of conventional moments. Obviously, these two methods were included in the experiment. The alternate method of fit, using Q_3 and Q_4, was also be included in the experiment. As Mykytka (1978) has shown, the method of alternate moments suffers less from sampling variability

effects than conventional moments. In addition, alternate moments use a similar method for computing lambda parameters. Like conventional moments and linear moments, it uses Powell's search algorithm to find the third and fourth lambda parameters. The objective function subroutine is the only difference between the three methods.

Although Chou (1988) showed that the method of maximum likelihood estimators (MLE's) produces better fits and better parameter estimates than conventional moments, it was not included in this experiment since its cpu-time requirements remain excessive for most applications.

5.5 Measures of Effectiveness

5.5.1 Statistical Summary. Table 5.2 is an example of the summary of information collected from each experiment. Each experiment consisted of fitting the GLD to thirty different data samples using each of the three estimation methods. The notes at the bottom of the table describe the experiment. The fourteen summary statistics listed along the left-hand side of the table were collected from each replication of the experiment. For each of the fourteen categories, the minimum and maximum values of each statistic over the thirty replications are displayed in the table. In addition, the mean and variance were calculated for the group of thirty values in each category and were also displayed in the table. The rows labelled "Theoretical K-S Statistics," and "Empirical K-S Statistics" contain information about the Kolmogorov–Smirnov statistics collected from comparisons of the theoretical CDF to the fitted CDF, and of the empirical CDF to the fitted CDF, respectively.

Experiment Result		Statistic			
		Minimum	Maximum	Average	Variance
Moments of Sample Data	1st	-0.32920000	0.33291600	-0.08192267	0.03075791
	2nd	0.46495600	0.69995600	0.56678553	0.00435006
	3rd	-0.14980300	0.16846000	0.02440728	0.00632024
	4th	-0.01568870	0.20792600	0.09713558	0.00315052
Lambda Parameters	λ_1	-2.07855000	1.39302000	-0.28595313	0.73005042
	λ_2	0.22071600	0.49963800	0.19608461	0.02915614
	λ_3	-0.08825540	1.00248000	0.18008392	0.06060880
	λ_4	-0.12274100	1.11750000	0.23446677	0.07488433
Theoretical K-S Statistics	MIN	0.00000211	0.00132966	0.00015583	0.00000008
	AVG	0.01000620	0.05861190	0.03572510	0.00016930
	MAX	0.02489270	0.18884700	0.10145870	0.00149412
Empirical K-S Statistics	MIN	0.00000000	0.00024583	0.00002845	0.00000000
	AVG	0.01290350	0.02903130	0.02055428	0.00002040
	MAX	0.06148730	0.14373100	0.09572504	0.00051787
Experiment Number: 1A					
Samples: 30 samples of 25 elements each					
Original function: Lambda approximation of a Normal function					
Method of fit: method of linear moments					

Table 5.2 Statistical summary of Monte Carlo experiment results

5.5.2 PDF's and Visual Fits. Figures 5.1, 5.2, and 5.3 are examples of the three distributions used in the experiment. Each figure contains both a plot of the theoretical density function and a plot of the density function estimated from a 1000-element sample by the method of linear moments. The experimental results in Appendices B through J contain plots of 25 estimated density functions from their respective sets of data samples. A compar-

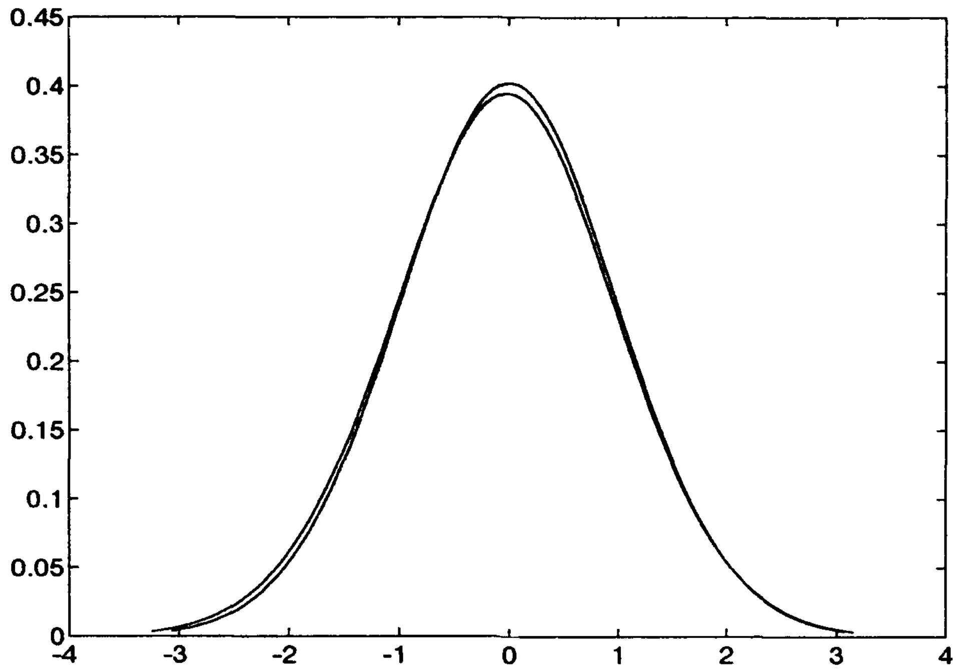

Figure 5.1 Comparison of the theoretical "Normal" PDF to the PDF fit from a sample of 1000 observations

ison of the different plots shows the methods' relative abilities to consistently estimate the parameters of the underlying distributions. These plots are suitable for visual comparisons only, and will be discussed in Chapter Six. No effort is made to collect quantitative data from these graphs.

5.5.3 CDF's and Kolmogorov–Smirnov Statistics. Figure 5.4 is an example of the plot of the CDF of the theoretical "Normal" distribution and a plot of the CDF of a distribution estimated from a 25-element sample by the method of linear moments. Figure 5.5 is an example of the plot of the empirical CDF of a 25-element sample and the CDF plot of the distribution estimated from that sample by the method of L-moments. From such information, various Kolmogorov–Smirnov goodness-of-fit statistics were computed. In particular, using the "Normal" distribution shown in Figure 5.4 as an example, the vertical distance

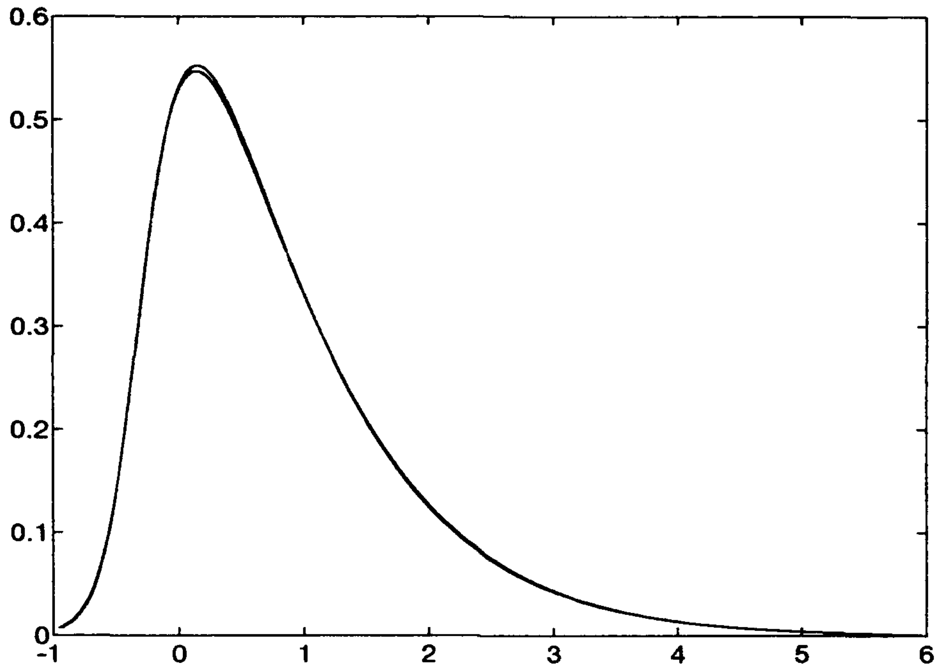

Figure 5.2 Comparison of the theoretical "Gamma" PDF to the PDF fit from a sample of 1000 observations

between the two CDFs were measured at 598 evenly–spaced points along the horizontal axis, beginning at -2.99 and ending at 2.99. From these measurements, the minimum distance, maximum distance, and the average distance were recorded. These distances are referred to as the minimum, maximum, and average K-S values. A comparison of a fitted CDF to a theoretical CDF attempts to capture the method's ability to estimate the underlying distribution of the population. The maximum Kolmogorov–Smirnov value typically represents a worst–case error between the fitted and theoretical CDF's. The average Kolmogorov–Smirnov value represents a measure of overall success in estimating a fit to the respective population or sample.

The results of applying these measures of fit to the results of the Monte Carlo experiment are described in Chapter Six.

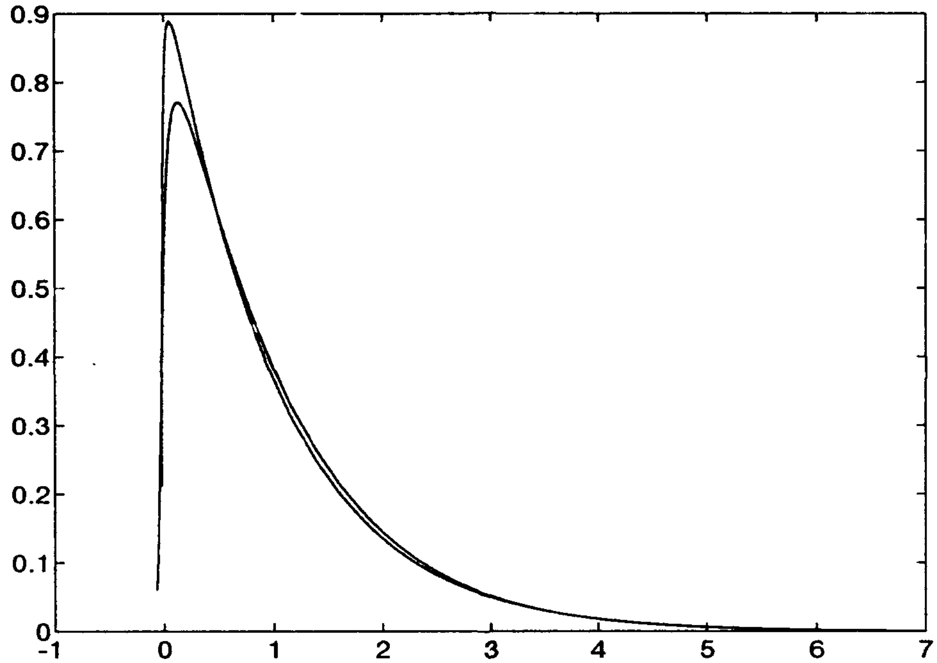

Figure 5.3 Comparison of the theoretical "Exponential" PDF to the PDF fit from a sample of 1000 observations

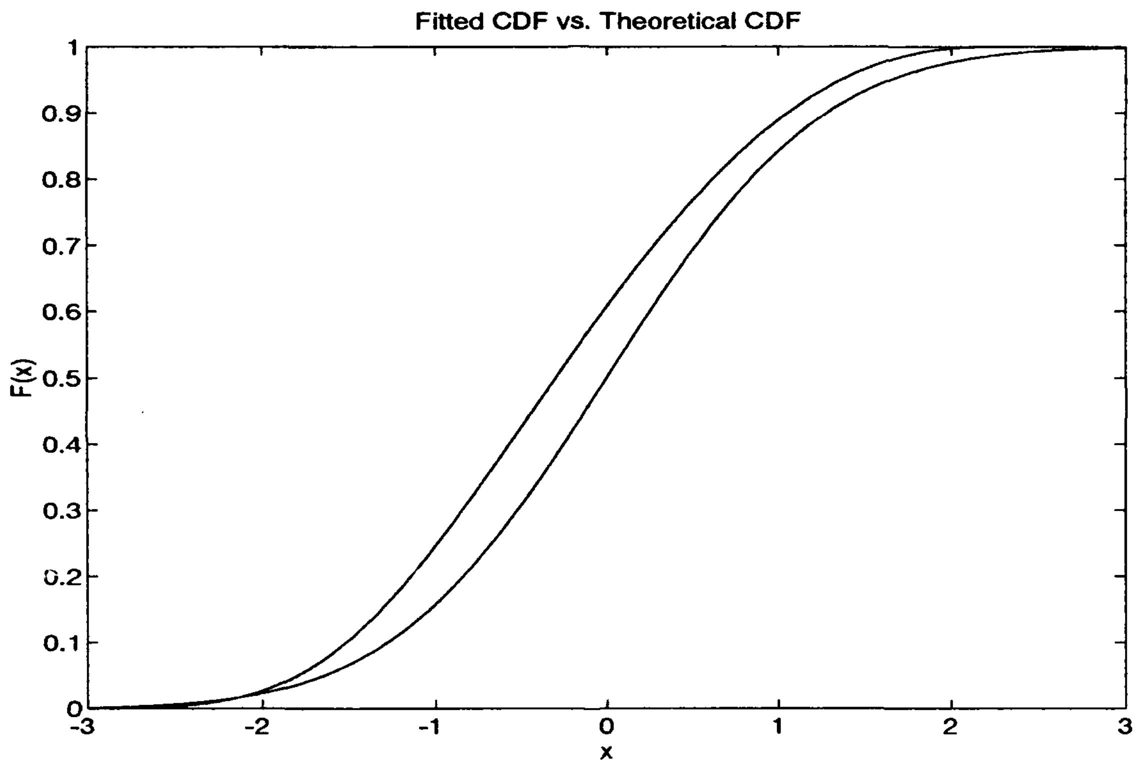

Figure 5.4 Comparison of the theoretical "Normal" CDF to the CDF fit from a sample of 25 observations

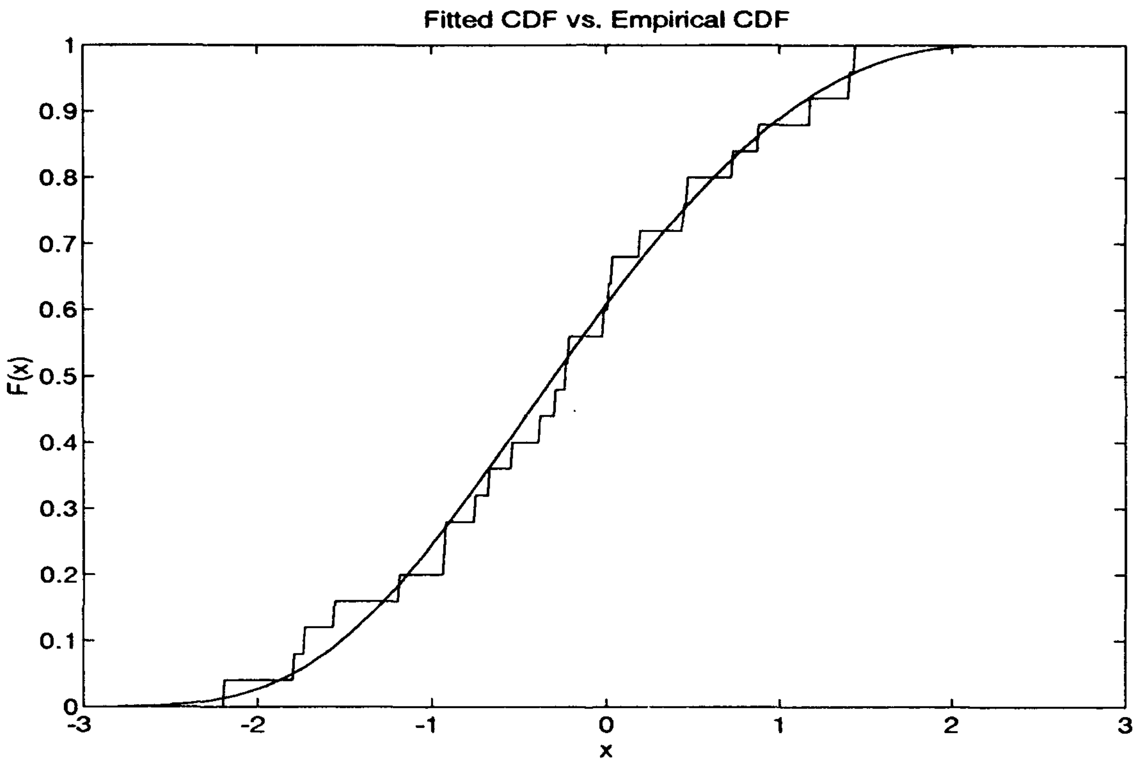

Figure 5.5 Comparison of the empirical CDF of, and a CDF fit to, a sample of 25 observations from the "Normal" distribution

VI. Results of the Monte Carlo Experiments

6.1 Tables of Experimental Results

Tables 6.1 through 6.10 and Figures 6.1 through 6.9 summarize the data collected from the Monte Carlo simulation study. The data is grouped to compare the three different methods of fit. The data from the experiments can be found in Appendices B through J.

6.2 Fitted Distributions vs. the Underlying Theoretical Distribution

6.2.1 Comparing Probability Density Function Plots. Figures 6.1 through 6.9 display the underlying theoretical probability density functions of the sample data used in the Monte Carlo experiment. Alongside the theoretical PDF are twenty-five examples of PDF's fitted to sample data by each of the three methods of moments. Appearing clockwise from the theoretical PDF are the linear, conventional, and alternate methods, respectively. The same data samples are used by each of the three estimation methods.

The comparison in Figure 6.1 of PDF's fitted from 25-element samples shows that a substantial amount of "sampling variability," or "noise," is transferred from the sample data to the fitted distributions by all three methods. Note that each set of examples contains two remarkably poor fits. An examination of the individual samples shows that the "poor fits" are caused by the same two samples in each case. Further, these two fits have the highest Kolmogorov-Smirnov statistics found in the entire set of thirty samples.

The comparison in Figure 6.2 of PDF's fitted from 50-element samples shows the noise carried over from the sample data is reduced by the larger sample size, but suggests that the different methods of fit remain basically indistinguishable. Again, note that each set contains one poorly-fitted distribution. Each is a result of the same data sample. That sample also produces the highest K-S value in the set of samples.

The comparison in Figure 6.3 of PDF's fitted from 100-element samples shows that the noise is substantially reduced by the sample size for all three methods of moments and that

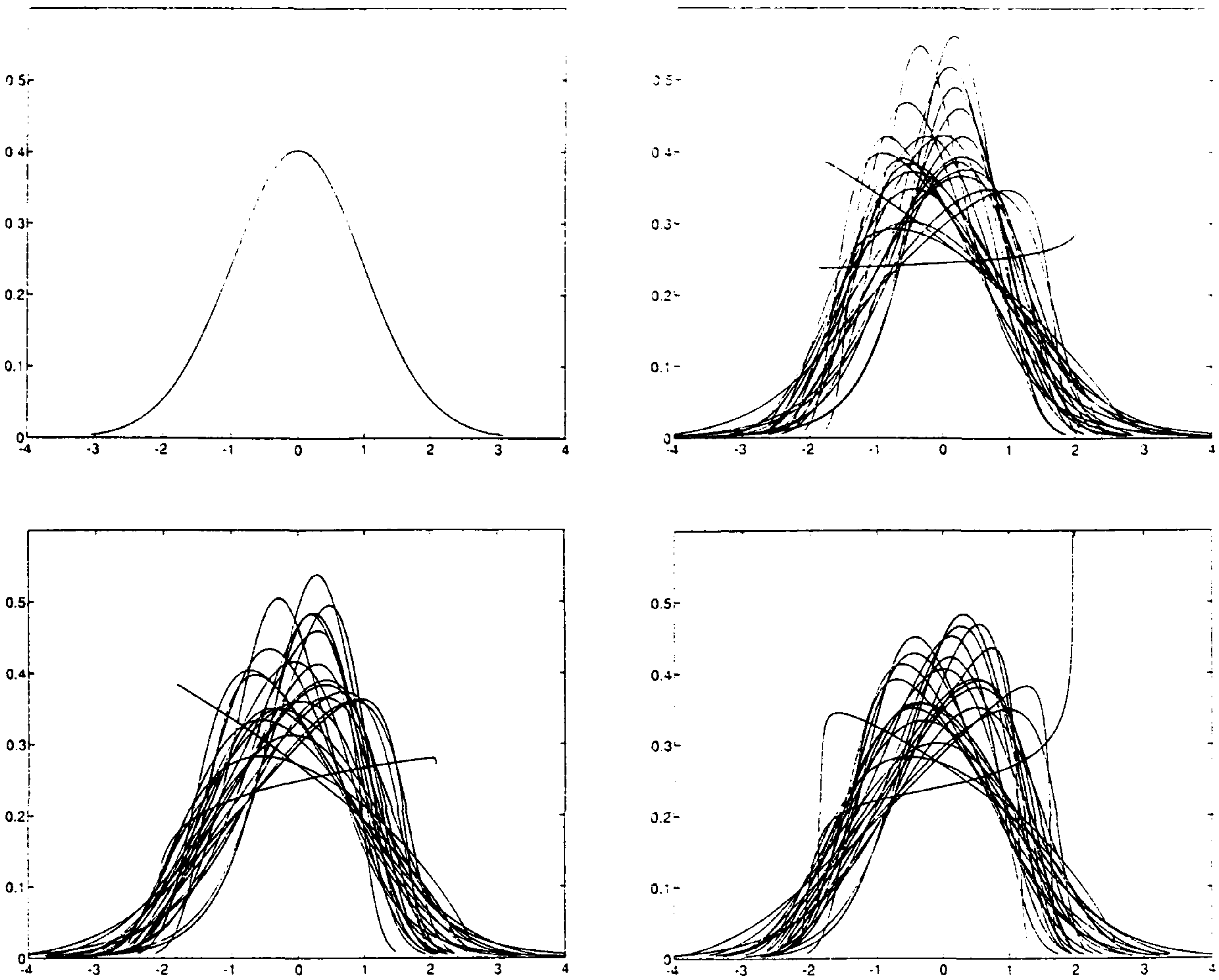

Figure 6.1 Theoretical "Normal" PDF vs. Methods of Fit of 25-Element Samples

there is no visible difference between groups of distributions fitted by the different methods. Also, this sample size was apparently sufficient to eliminate any radically-different fits caused by sampling variability.

The comparison in Figure 6.4 of PDF's fitted from 25-element samples selected from the "Gamma" distribution shows that a significant amount of noise is again carried to the estimated PDF's by all three methods. Note that the conventional method of moments has noticeably more "poor fits" than the other two methods. Several fits have taken on an

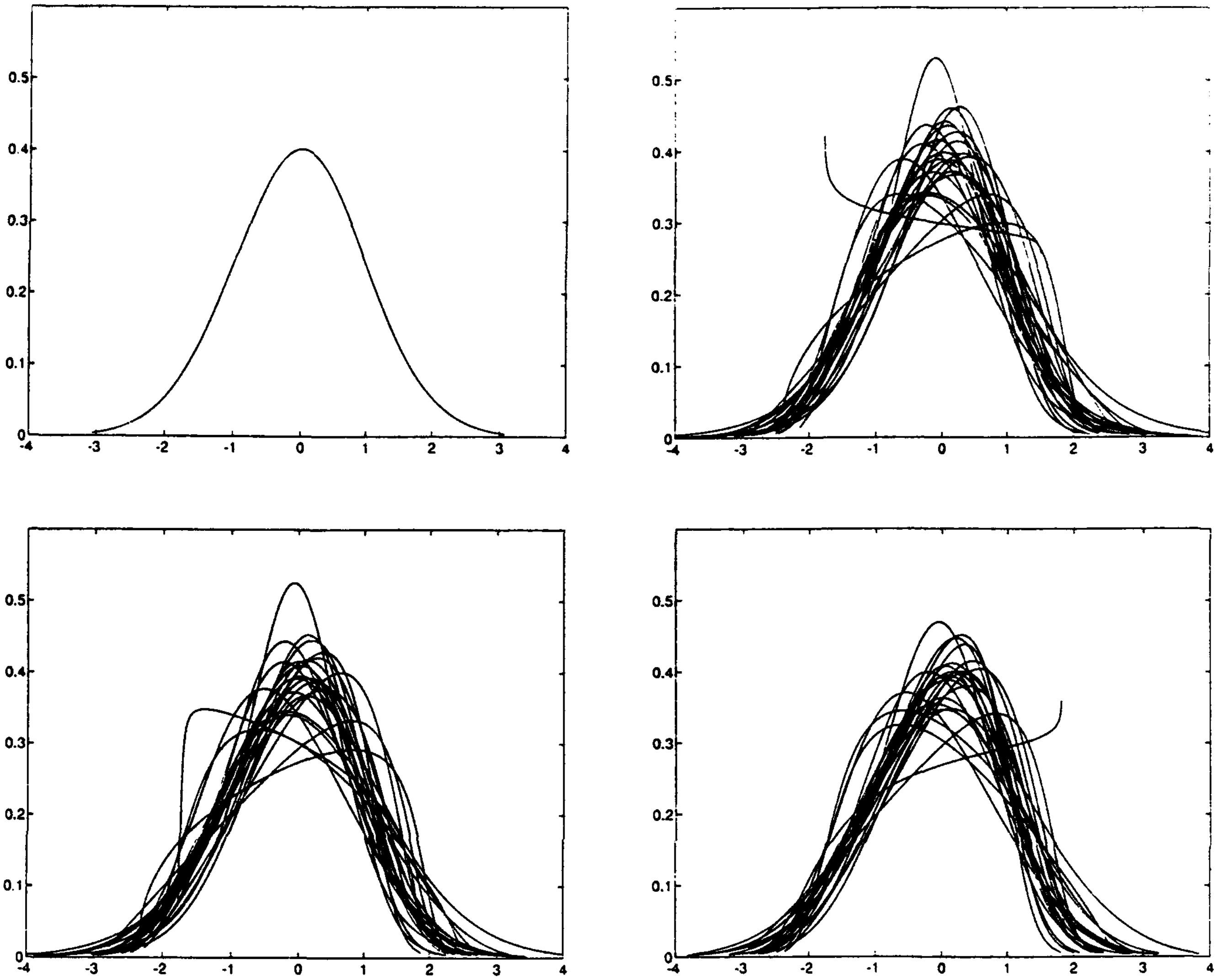

Figure 6.2 Theoretical "Normal" PDF vs. Methods of Fit of 50-Element Samples

Exponential-like "ski slope" shape, rather than the desired skewed unimodal shape of the "Gamma" distribution.

The comparison in Figure 6.5 of PDF's fitted from 50-element samples shows that the three methods again exhibit roughly equal levels of sampling variability, and that the effects of sampling variability are reduced by the increase in sample size.

The comparison in Figure 6.6 of PDF's fitted from 100-element samples shows that the increase in sample size continues to reduce the effects of sampling variability for all three

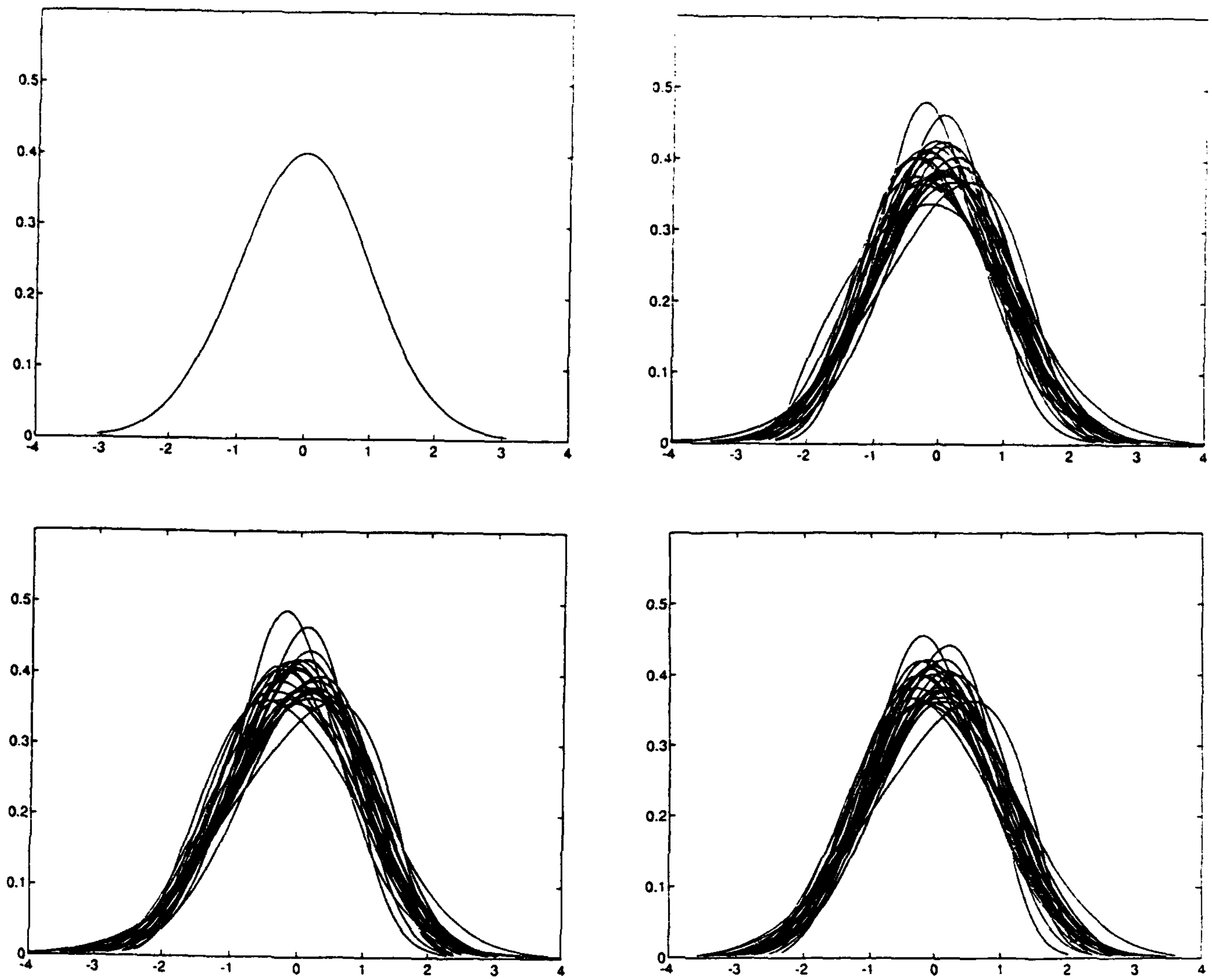

Figure 6.3 Theoretical "Normal" PDF vs. Methods of Fit of 100-Element Samples

methods of fit. Note, however, that this sample size no longer eliminates "poor fits" by conventional moments.

The comparison in Figure 6.7 of PDF's fitted from 25-element samples shows that, for the "Exponential" case, the different methods of fit show different levels of resistance to the effects of sampling variability. The conventional moments show the least resistance, with the largest group of "Exponential" shaped distributions, while the linear moments show the most resistance, with the "tightest" group of "Exponential" shaped distributions.

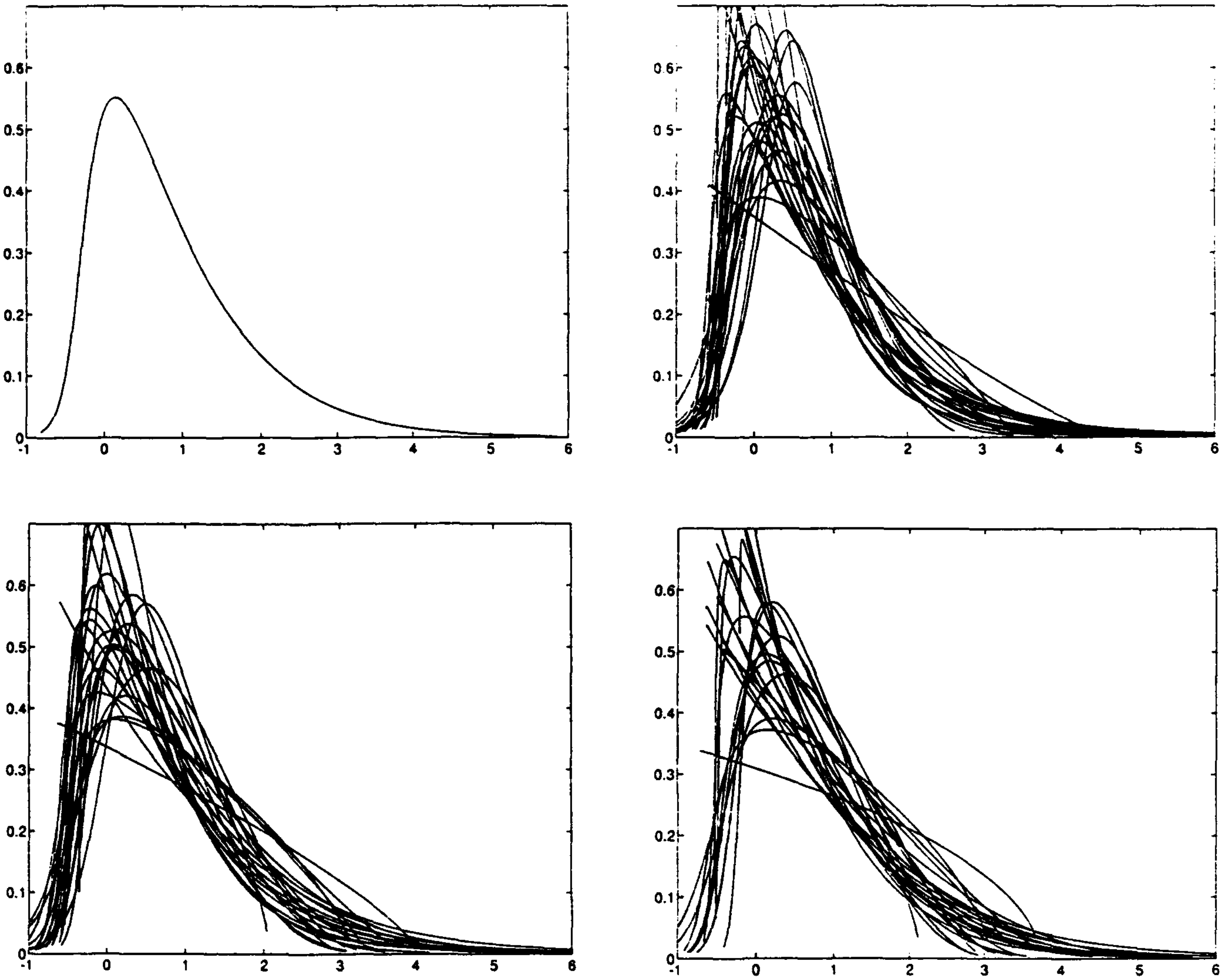

Figure 6.4 Theoretical "Gamma" PDF vs. Methods of Fit of 25-Element Samples

The comparison in Figure 6.8 of PDF's fitted from 50-element samples shows that the sample size has reduced the effective noise of the sample, but the relative "behavior" of the different methods shown in the previous Figure still remains in effect.

The comparison in Figure 6.9 of PDF's fitted from 100-element samples shows that this sample size is insufficient in this case to mask the noise in sample data from any of the methods of fit. This suggests that as the skewness and kurtosis of an underlying distribution increases, the size of sample data must also increase to mask a given level of sampling variability. The relative abilities of the different methods to reduce the sampling variability

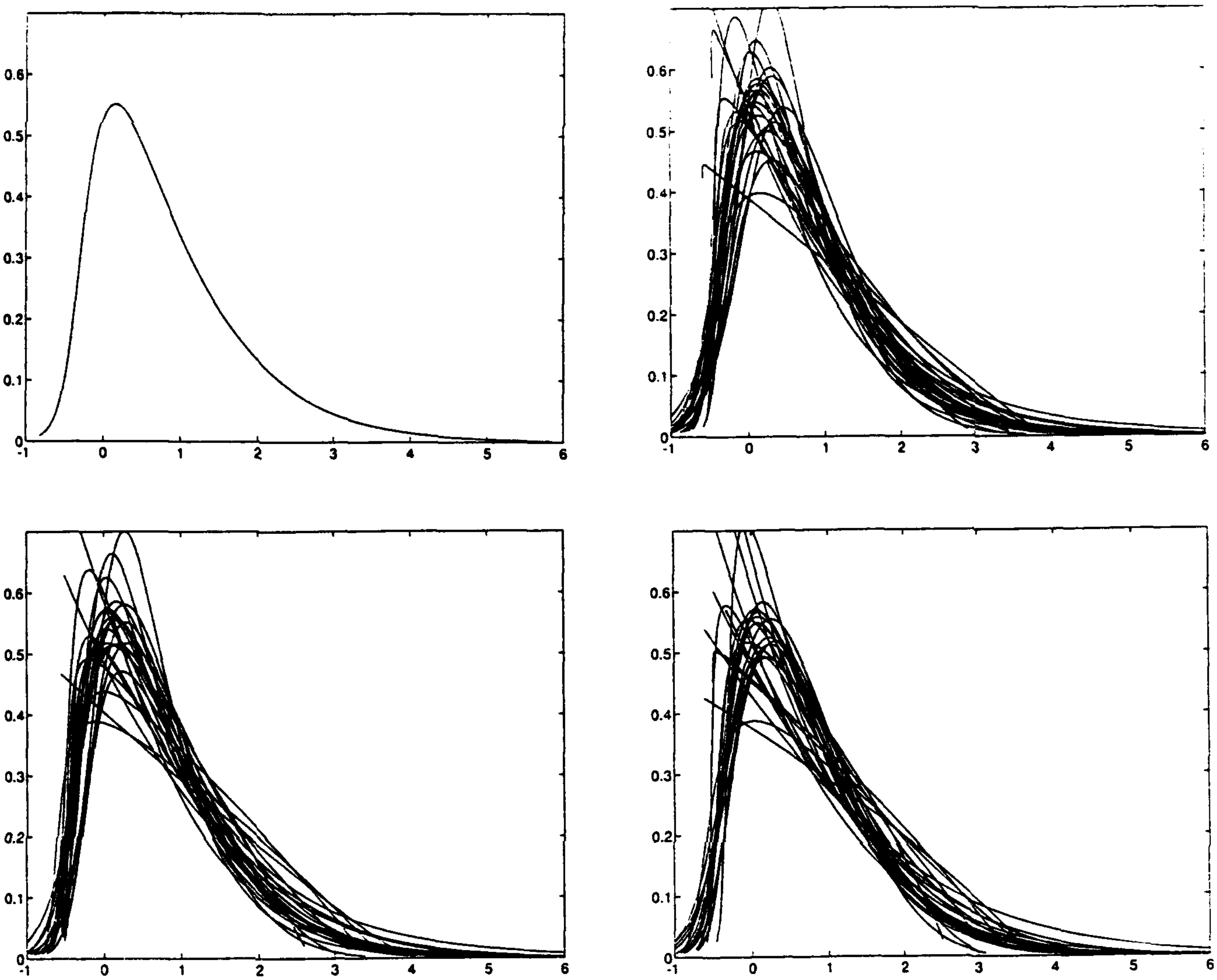

Figure 6.5 Theoretical "Gamma" PDF vs. Methods of Fit of 50-Element Samples

effects remains visible. The linear moments remain most effective, while the conventional moments are still least effective.

This visual comparison of groups of fitted distributions does provide a subjective feel for which methods of fit are better or worse, but it does not provide an objective or more rigorous comparison. In order to make a quantitative assessment of the relative differences in the methods of fit, an effort was made to determine the percentage of fitted distributions which accurately capture the shape of the underlying distribution.

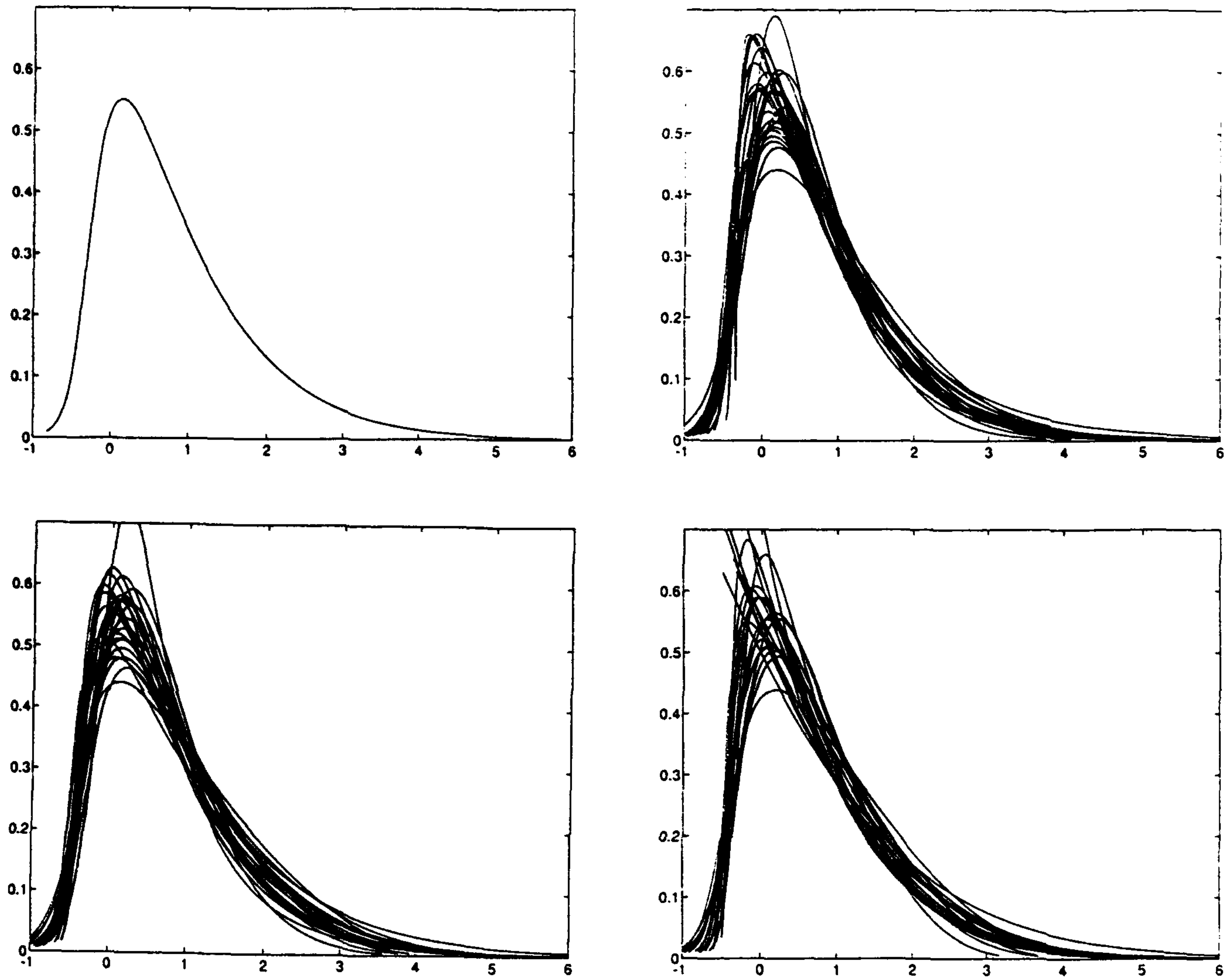

Figure 6.6 Theoretical "Gamma" PDF vs. Methods of Fit of 100-Element Samples

6.2.2 Summarizing Probability Density Function Plots. Table 6.1 summarizes the observations made from the visual comparisons of the PDF's estimated from the Monte Carlo simulation study and lists the number of "poor fits" observed in each set of 25 PDF's estimated by the linear, conventional, and alternate methods, respectively. Any plotted curve that does not resemble the shape of the underlying distribution constitutes a poor fit. Figures 6.10 through 6.12 provide examples of fitted PDF's which do not resemble the shapes of the theoretical "Normal," "Gamma," and "Exponential" PDF's, respectively.

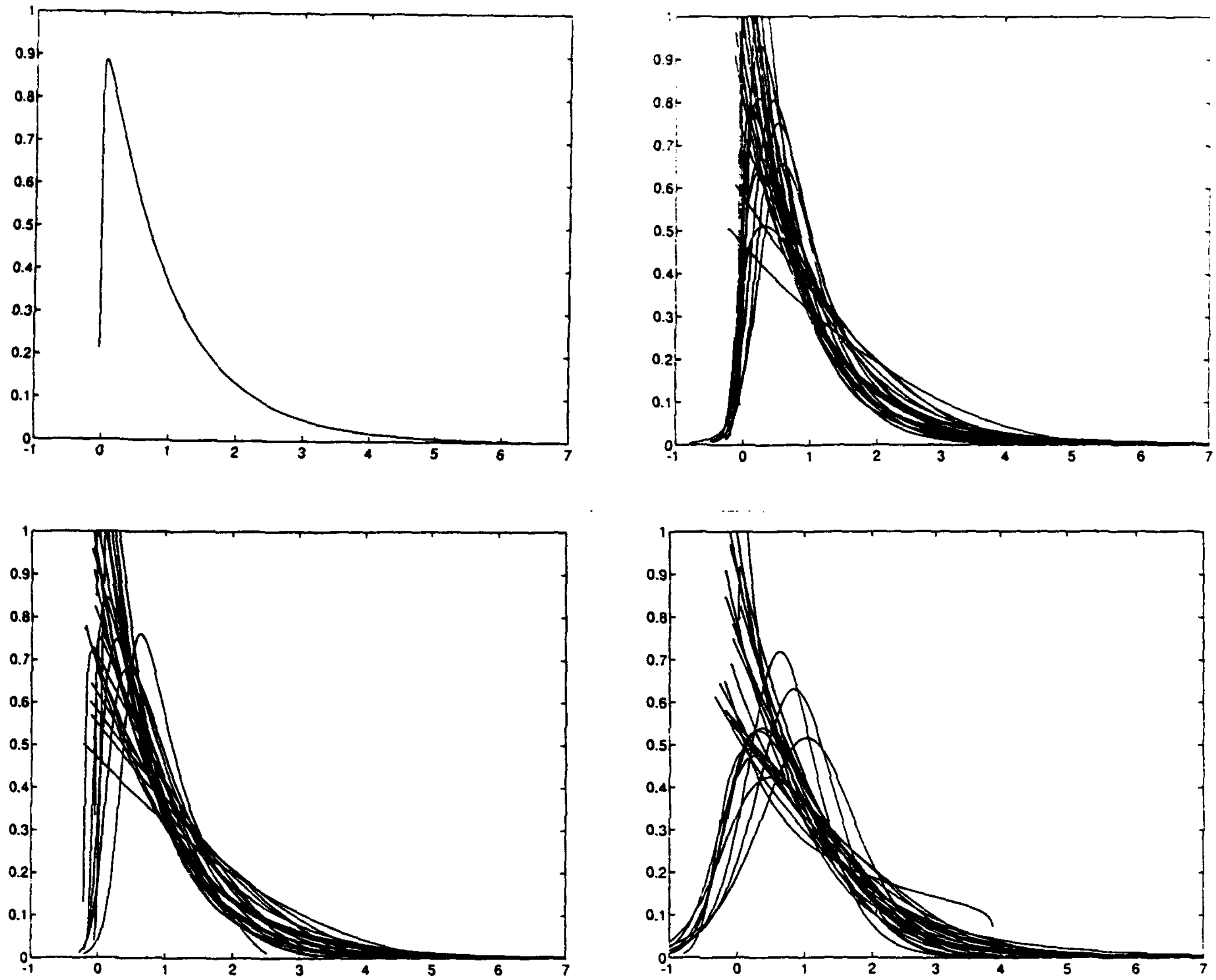

Figure 6.7 Theoretical "Exponential" PDF vs. Methods of Fit of 25-Element Samples

A distribution fitted from "Normal" distributed sample data would be considered a "poor fit" if it was not unimodal, roughly symmetrical, and approximately "bell-shaped." A distribution fitted from "Gamma" distributed sample data would be considered a "poor fit" if it was not unimodal, asymmetrical, and skewed to the left. A distribution fitted from "Exponential" distributed sample data would be considered a "poor fit" if it was not concave, asymmetrical and skewed to the left, and approximately "ski slope-shaped."

Table 6.1 shows that all three methods of fit are very capable of estimating good fits to the "Normal" distribution, at any of the tested sample sizes. The methods of linear moments

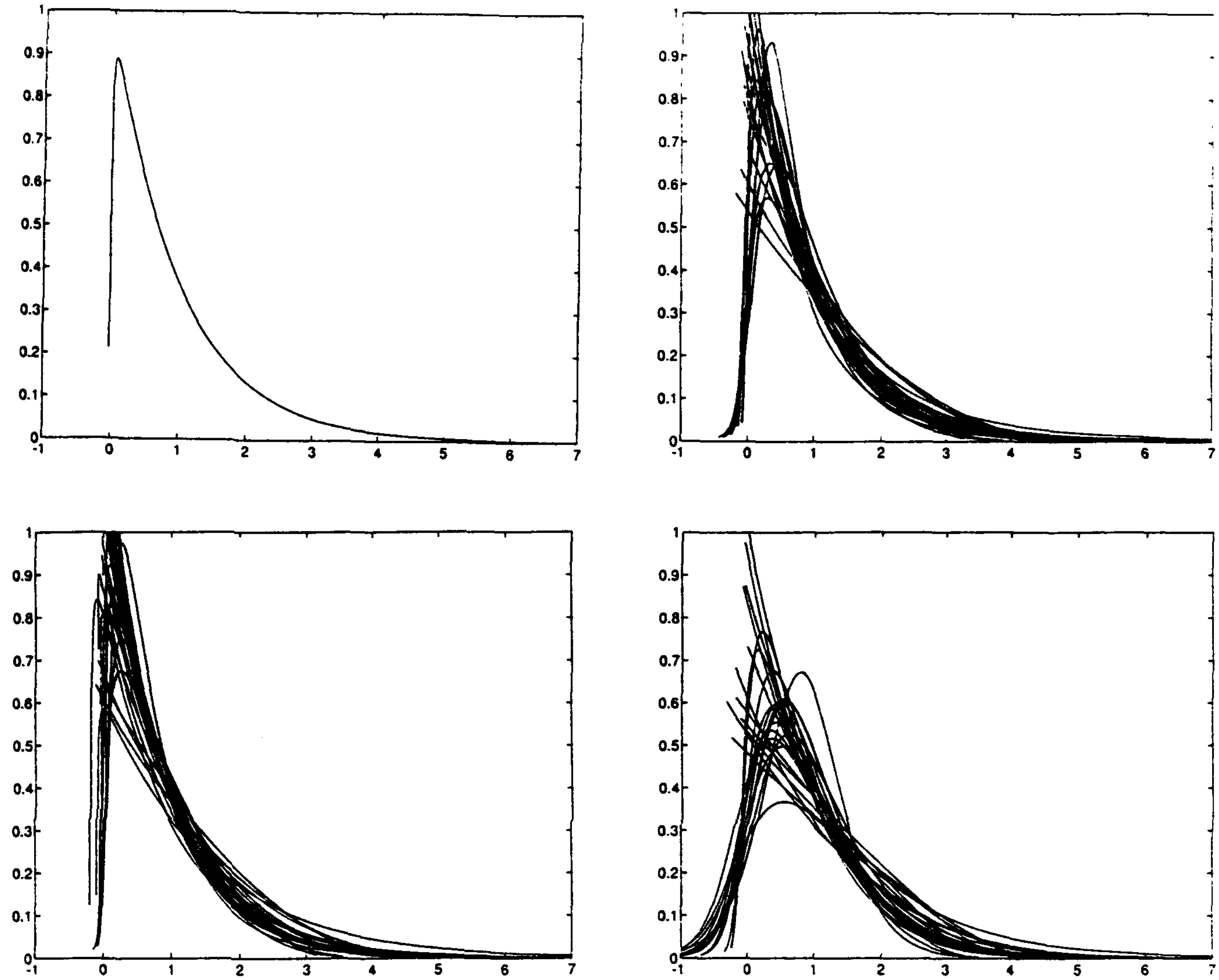

Figure 6.8 Theoretical "Exponential" PDF vs. Methods of Fit of 50-Element Samples

and alternate moments were noticeably better than conventional moments at estimating the PDF's of samples from the "Gamma" distribution. While the "Exponential" distribution was the most difficult to estimate for all three methods, it was particularly difficult for the method of conventional moments. That method was effectively unreliable for estimating a good fit from an "Exponential" distributed sample. The method of linear moments was better than the method of alternate moments, but not to the degree of its advantage over the method of conventional moments. Also, for a given sample size, sample data from

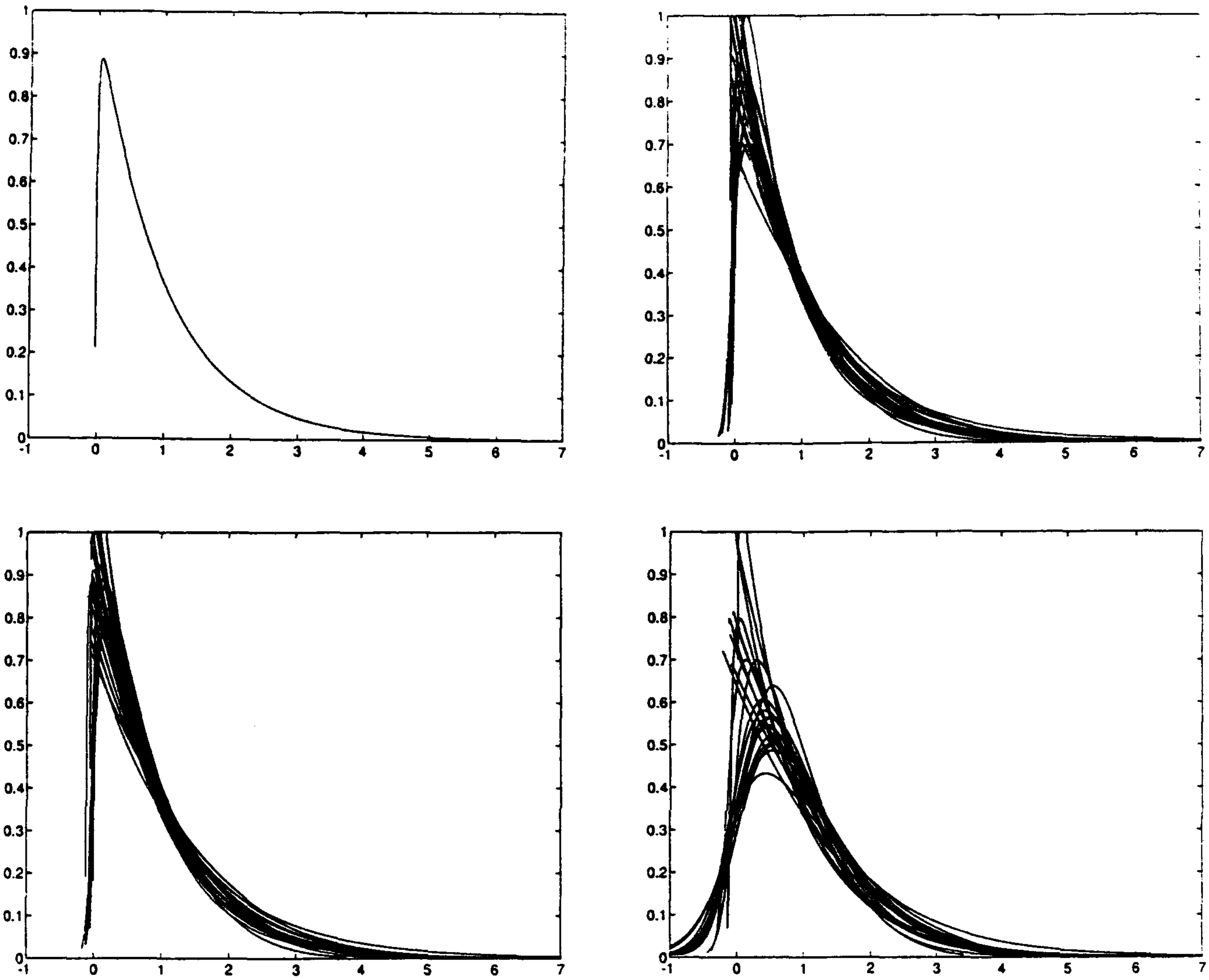

Figure 6.9 Theoretical "Exponential" PDF vs. Methods of Fit of 100-Element Samples

distributions with higher skewness and kurtosis values transferred more noise to the fitted distributions through all three methods of moments.

Although this summary of the fitted PDF's provides an intuitive feel for the relative advantages or disadvantages of the various methods of moments, it is still rather subjective. In addition, while the assessments made so far are useful in comparing different methods to each other, they are not so useful for comparing them against the theoretical distribution itself.

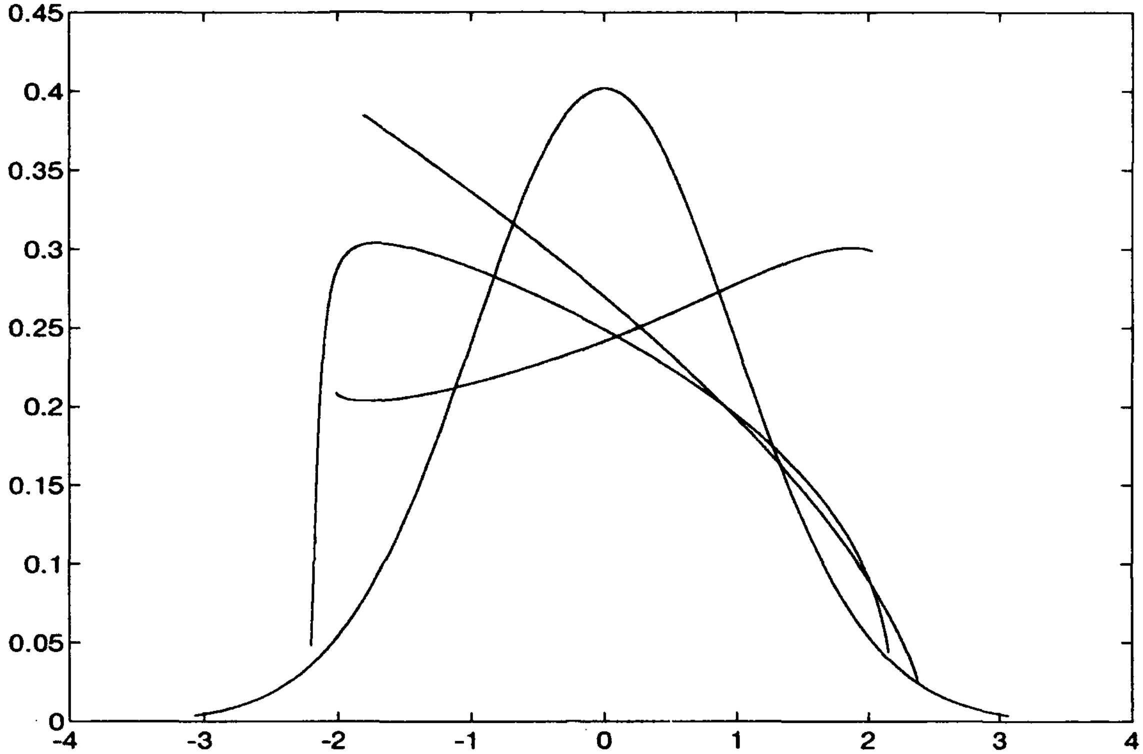

Figure 6.10 Examples of "poor fits" to the theoretical "Normal" PDF

The Kolmogorov–Smirnov goodness–of–fit test is a quantitative measure of the accuracy with which a fitted CDF matches the underlying CDF. Various K-S statistics were collected from the fitted CDF of every data sample and compiled to provide a quantitative measure of the goodness–of–fit of each of the three methods of moments, both for comparison to the theoretical distribution, and to each other.

6.2.3 Summarizing the Kolmogorov–Smirnov Statistics. Table 6.2 summarizes the Kolmogorov–Smirnov (K-S) statistics collected from each of the 27 Monte Carlo experiments. The "Average K-S" Column lists the average over the 30 samples of the average difference between the estimated and theoretical distribution functions. The "Maximum K-S" Column lists the average over the 30 samples of the maximum difference between the same functions. The "Max K-S Variance" Column lists the variance of the 30 values which form the average in previous column.

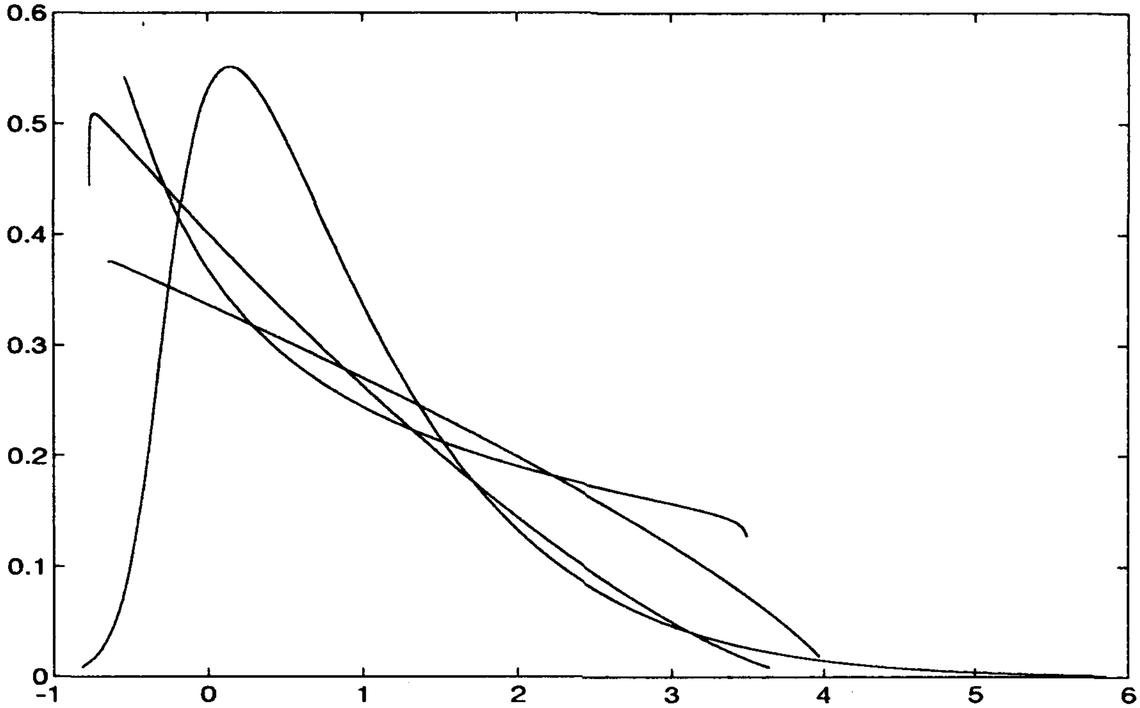

Figure 6.11 Examples of "poor fits" to the theoretical "Gamma" PDF

When comparing the different methods of moments for like sample distributions and sample sizes, there is little difference among the averages of the K-S statistics. The K-S averages of the linear and alternate methods are identical to one significant digit, and in most cases, identical to two significant digits. The conventional method is only slightly worse for the "Gamma" and "Exponential" distributions. Based on these averages, no method shows an obvious advantage over another.

However, there are noticeable differences in the variances of these sample sets among the different methods of fit. For the "Normal" distribution, all three methods perform equally well for all sample sizes. But for the "Gamma" and "Exponential" distributions, particularly for the smaller sample sizes, the linear moments and alternate moments achieve smaller variances than the conventional moments. Both methods average a noticeable decrease in variance over the conventional moments for all sizes of "Gamma" and "Exponential"

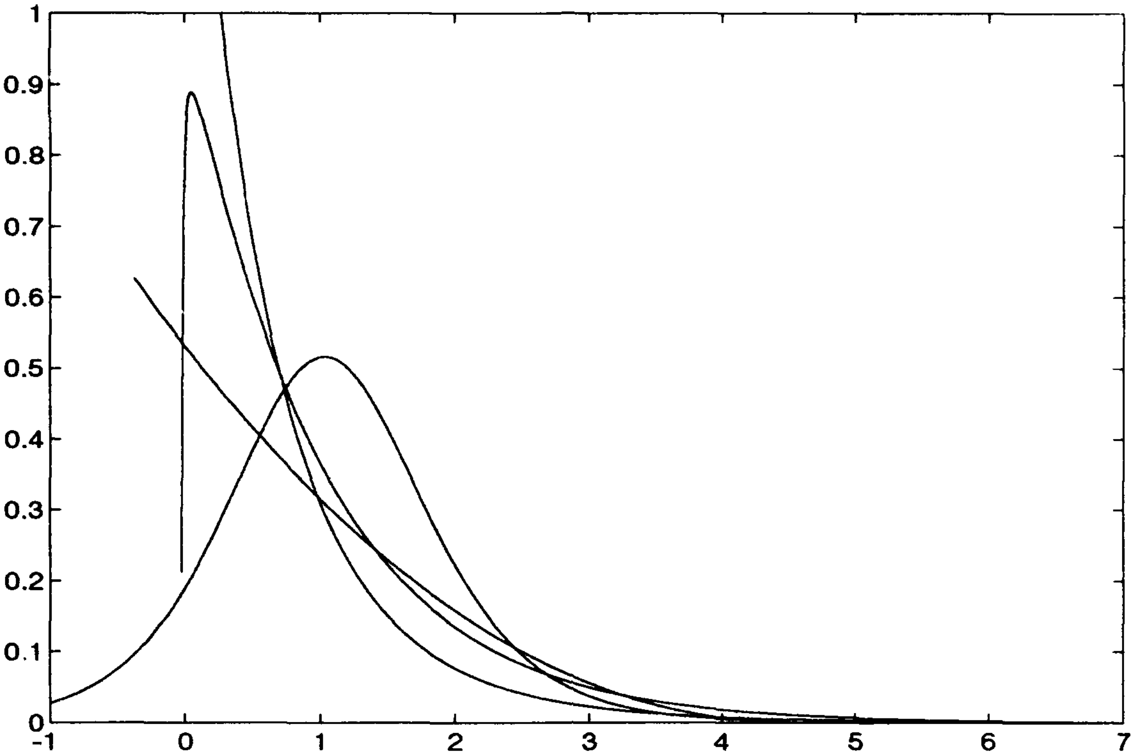

Figure 6.12 Examples of "poor fits" to the theoretical "Exponential" PDF

distributed samples. This reduced variance shows that the methods of linear and alternate moments produce more consistent estimates for skewed distributions.

Number of "Poor Fits" out of 25 Samples				
Distribution Shape	Sample Size	Linear Moments	Conventional Moments	Alternate Moments
"Normal"	25	2	3	3
	50	1	1	1
	100	0	1	0
"Gamma"	25	2	14	4
	50	3	9	3
	100	0	7	0
"Exponential"	25	8	12	7
	50	9	15	6
	100	4	14	3

Table 6.1 Summary of PDF Plots

Distribution Shape	Sample Size	Method of Moments	Average K-S	Maximum K-S	Max K-S Variance
"Normal"	25	linear	0.03573	0.10150	.00149
		conventional	0.03548	0.09837	.00104
		alternate	0.03524	0.09806	.00119
	50	linear	0.02452	0.06728	.00086
		conventional	0.02522	0.06705	.00100
		alternate	0.02454	0.06675	.00087
	100	linear	0.01886	0.05100	.00051
		conventional	0.01836	0.04879	.00048
		alternate	0.01863	0.05116	.00056
"Gamma"	25	linear	0.02790	0.09577	.00132
		conventional	0.02837	0.09355	.00104
		alternate	0.02824	0.09368	.00099
	50	linear	0.02040	0.06530	.00077
		conventional	0.02319	0.07323	.00092
		alternate	0.02088	0.06534	.00081
	100	linear	0.01549	0.05020	.00044
		conventional	0.01741	0.05959	.00090
		alternate	0.01555	0.04911	.00045
"Exponential"	25	linear	0.02480	0.09237	.00128
		conventional	0.02693	0.11733	.00156
		alternate	0.02529	0.08874	.00174
	50	linear	0.01895	0.06403	.00063
		conventional	0.02239	0.09521	.00112
		alternate	0.01868	0.05966	.00082
	100	linear	0.01450	0.04547	.00038
		conventional	0.01663	0.07962	.00096
		alternate	0.01428	0.04437	.00043

Table 6.2 K-S Statistic Summary of fitted CDF's vs. the theoretical CDF

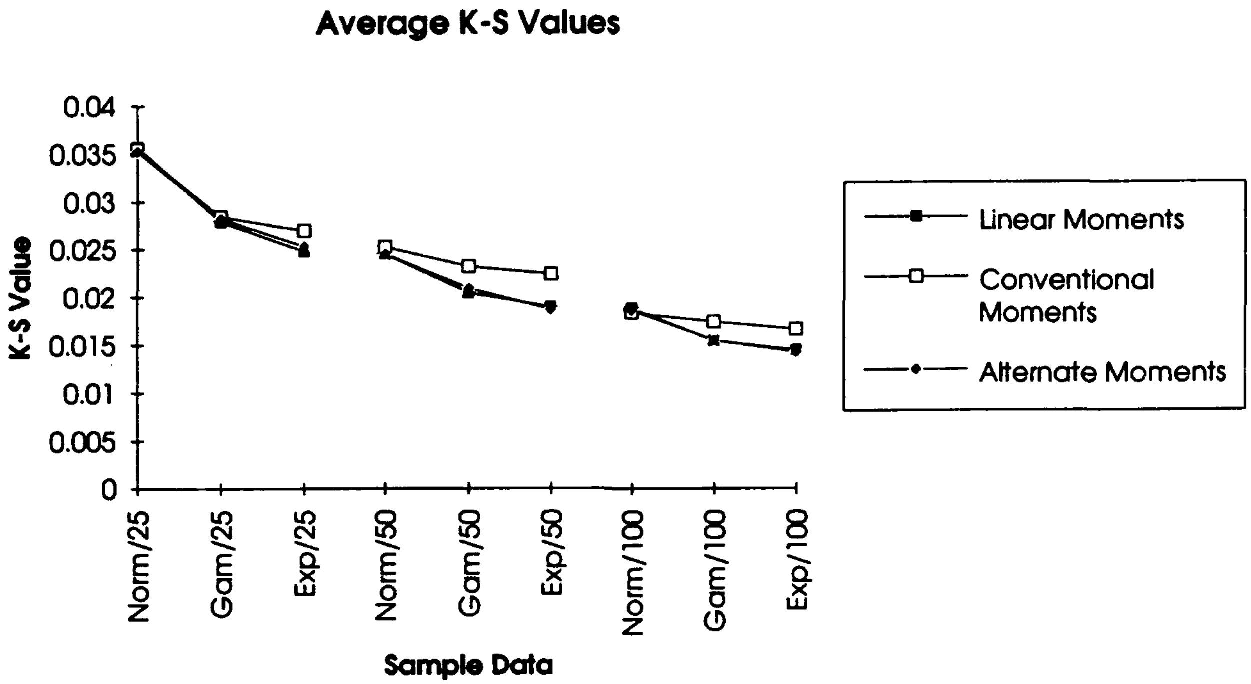

Figure 6.13 Average K-S Values of fitted CDF's vs. the theoretical CDF

Figures 6.13 through 6.15 display the information from the "Average K-S," "Maximum K-S," and "Max K-S Variance" columns of Table 6.2, respectively. These are helpful in highlighting the differences between the methods.

Figure 6.13 clearly shows that the linear and alternate moments are comparable while the conventional moments produce slightly higher averages of the K-S values. There are two downward trends depicted in the chart. The average K-S values decrease as the sample size increases, and as the skewness and kurtosis of the theoretical distribution increase.

Figure 6.14 shows even more disparity between the performance of the method of conventional moments and the other two methods. Note that for all cases of "Normal" distributed sample data, all three methods perform equally. For "Gamma" and "Exponential" distributed sample data, the K-S values associated with the conventional method of moments

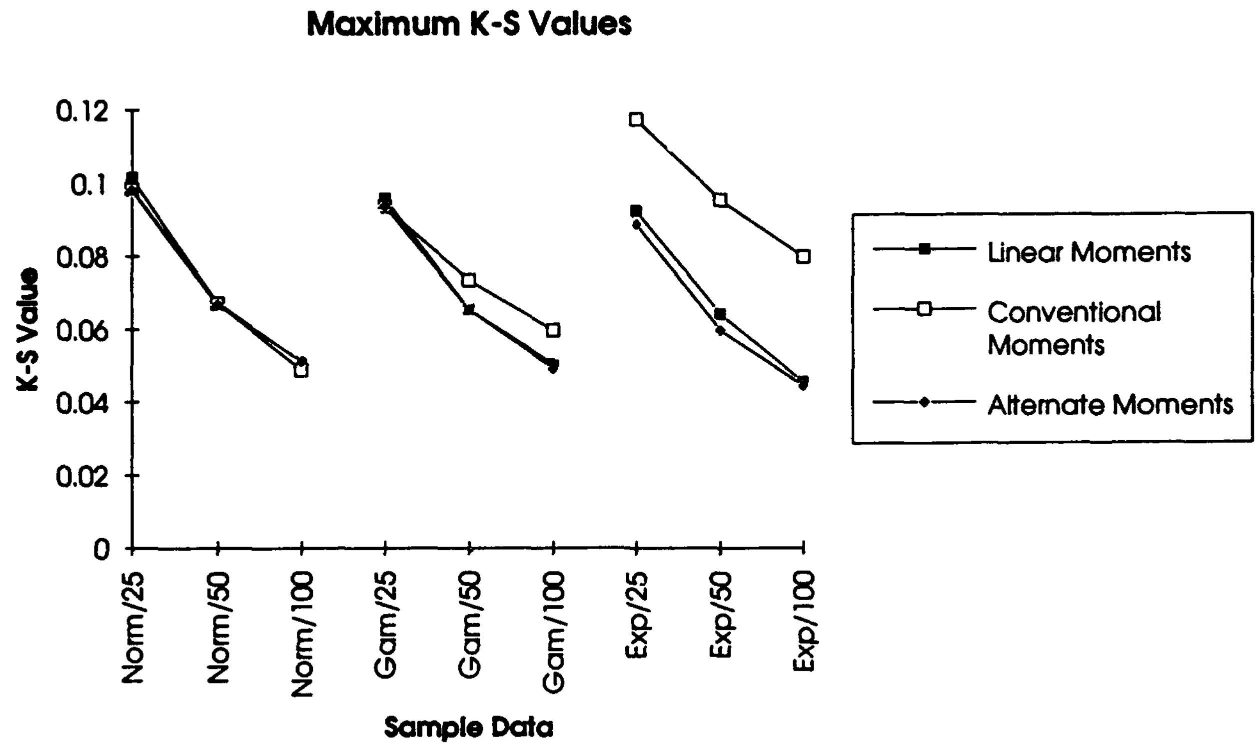

Figure 6.14 Maximum K-S Values of fitted CDF's vs. the theoretical CDF

become increasingly larger than those for the other moments as the skewness and kurtosis increases.

Figure 6.15 suggests some instability in the methods at low sample sizes as indicated by by the unusual behavior shown in the chart for 25-element sample data. For sample sizes of fifty elements and up, the previous trends continue. That is, the linear and alternate methods perform about equally, while the conventional moments do not perform as well. The variance in K-S values also tends to decrease as the sample size and skewness and kurtosis of the distribution increase.

Table 6.2 and Figures 6.13 through 6.15 provide a quick overall picture of the objective measures of the goodness-of-fit for each of the three methods of fit. The mean and variance of the K-S statistics provide an idea of which method will perform well in the long run. However, information on which method will most often provide the closest fit is also useful.

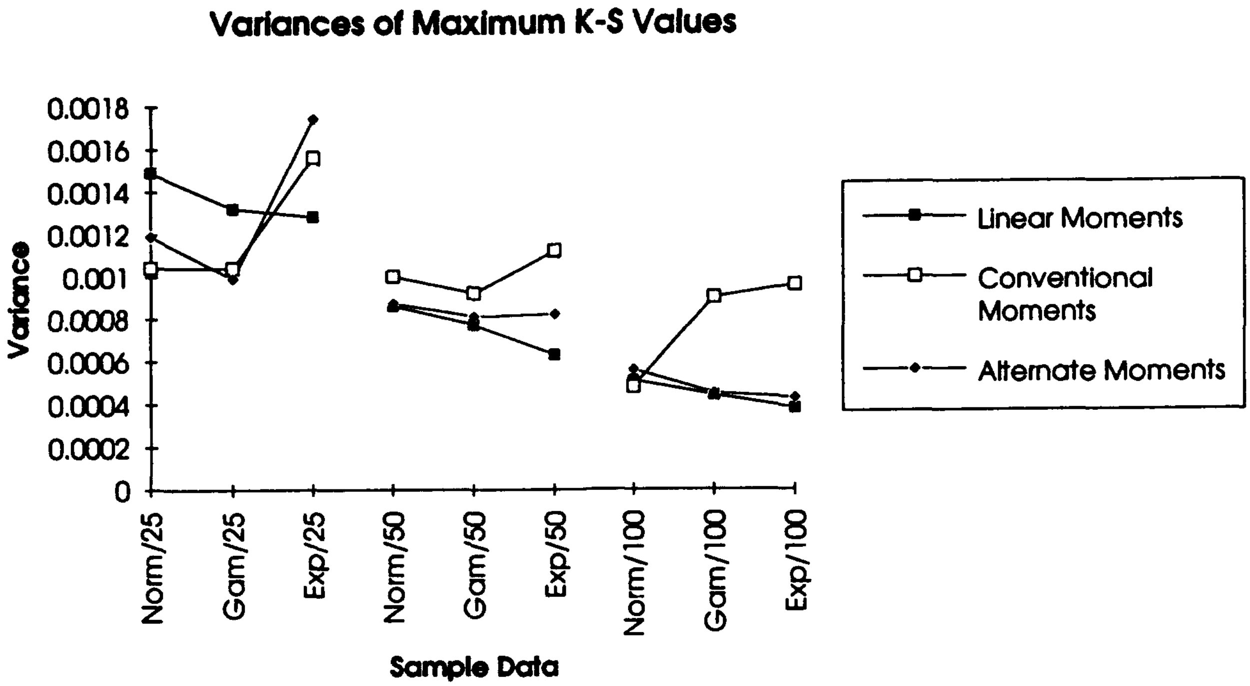

Figure 6.15 Variances of Maximum K-S Values of fitted CDF's vs. the theoretical CDF

6.2.4 Kolmogorov–Smirnov Statistics of Individual Data Samples. Tables 6.3 and 6.4 list the number of times that each method of fit scored the lowest K-S value among each in a set of thirty samples. The tables are divided into summaries of maximum and average K-S values. Each table is further broken down by the underlying distribution and size of the samples in the set.

Fitted Distribution vs. Theoretical Distribution				
Distribution Shape	Sample Size	MAX K-S of Linear Method	MAX K-S of Conventional Method	MAX K-S of Alternate Method
"Normal"	25	9	12	9
	50	8	12	10
	100	14	12	4
"Gamma"	25	13	7	10
	50	13	8	9
	100	13	6	11
"Exponential"	25	11	7	12
	50	11	4	15
	100	16	3	11

Table 6.3 Number of Lowest MAX K-S Theoretical Differences per set of 30 samples

Table 6.3 shows that the method of conventional moments most often provides the best fit to the underlying theoretical "Normal" distribution. The method of linear moments most often produces the closest fit to the underlying theoretical "Gamma" distribution. Both the linear and alternate moments are good for estimating the underlying theoretical "Exponential" distribution. None of the methods show a distinct superiority as the sample size is reduced. The lack of a consistently superior method suggests a lack of correlation between the method of fit used on sample data and the quality of fit in general.

Fitted Distribution vs. Theoretical Distribution				
Distribution Shape	Sample Size	AVG K-S of Linear Method	AVG K-S of Conventional Method	AVG K-S of Alternate Method
"Normal"	25	10	8	12
	50	8	13	9
	100	14	10	6
"Gamma"	25	15	7	8
	50	17	6	7
	100	14	7	9
"Exponential"	25	12	12	6
	50	11	9	10
	100	12	6	12

Table 6.4 Number of Lowest AVG K-S Theoretical Differences per set of 30 samples

Table 6.4 shows that the methods of linear and conventional moments estimate the theoretical "Normal" distribution well. The method of linear moments is best at fitting both the theoretical "Gamma" and "Exponential" distribution shapes. Again, no method shows a clear superiority for smaller sample sizes across all distributions (although L-moments look at least as good as the others).

Having examined the distributions visually and having looked at quantitative goodness–of–fit data taken from the distributions, another source of information is the set of lambda parameters estimated from the sample data. Comparing the parameter estimates to their theoretical values provides another measure of a method's ability to determine a distribution from sample data.

6.2.5 Summarizing Lambda Parameter Estimates. Table 6.5 lists the lambda parameter estimations of the "Normal" distributed data samples from the Monte Carlo simulation study. The first row contains the lambda parameters from the underlying theoretical distribution. The next nine rows contain the average parameter estimates from the thirty samples of the specified set. The last nine rows contain the variances of those thirty sample estimates.

"Normal" Distribution					
Fitment Method	Sample Size	λ_1	λ_2	λ_3	λ_4
Theoretical	∞	0	0.1975	0.1349	0.1349
Average of	25	-.2859	0.1961	0.1801	0.2345
Linear	50	0.0269	0.2368	0.2386	0.2032
Moments	100	-.0547	0.2340	0.1752	0.1802
Average of	25	-.1466	0.2427	0.2055	0.2544
Conventional	50	0.0692	0.2506	0.2197	0.2109
Moments	100	-.0462	0.2515	0.1884	0.2047
Average of	25	0.0149	0.2143	0.2245	0.1853
Alternate	50	-.0303	0.2288	0.1974	0.2061
Moments	100	-.0472	0.2370	0.1801	0.1912
Variance of	25	0.7301	0.0292	0.0606	0.0749
Linear	50	0.3613	0.0160	0.0586	0.0469
Moments	100	0.1271	0.0061	0.0099	0.0098
Variance of	[illegible]	0.8896	0.0051	0.0271	0.0726
Conventional	50	0.3456	0.0041	0.0139	0.0474
Moments	100	0.1484	0.0043	0.0056	0.0181
Variance of	25	0.7830	0.0166	0.0473	0.0407
Alternate	50	0.5484	0.0079	0.0276	0.0442
Moments	100	0.1759	0.0065	0.0089	0.0183

Table 6.5 Lambda Parameter Estimates of "Normal" Distributed Samples

Comparing the theoretical values to the estimated sample averages shows that the method of linear moments usually produces the most accurate estimates. It is also apparent that an increase in sample size does not necessarily produce an increase in accuracy. As

for the variances of the sample sets, the method of linear moments is comparable to the method of conventional moments in consistency of estimation. Both are marginally better than alternate moments. In all cases, an increase in sample size produces a decrease in variance, suggesting an increase in the consistency of parameter estimation for all methods, as the sample size increases.

Table 6.6 lists the lambda parameter estimations of the "Gamma" distributed data samples from the Monte Carlo simulation study. In this case, the method of linear moments again produces the most accurate estimates. The averages of the estimated lambda parameters are closer to the theoretical values than are the averages of the other two methods. As for consistency, the results are less conclusive. The lowest variances are scattered randomly among the various methods and sample sizes. Overall, the method of conventional moments maintains the lowest average variance. The only trend is that the variance continues to decrease as the sample size increases.

“Gamma” Distribution					
Fitment Method	Sample Size	λ_1	λ_2	λ_3	λ_4
Theoretical	∞	0	.04134	.00567	.04046
Average of Linear Moments	25	-.1298	0.0246	-.0011	0.0840
	50	-.1054	0.0876	0.0058	0.1323
	100	-.0853	0.0784	0.0083	0.0942
Average of Conventional Moments	25	-.3565	0.1927	0.0132	0.2916
	50	-.2512	0.1710	0.0112	0.2477
	100	-.2212	0.1362	0.0092	0.1760
Average of Alternate Moments	25	-.2003	0.1361	0.0158	0.2065
	50	-.1680	0.1308	0.0091	0.1878
	100	-.0949	0.0935	0.0096	0.1180
Variance of Linear Moments	25	0.0948	0.0522	0.0012	0.0398
	50	0.0659	0.0187	0.0004	0.0283
	100	0.0222	0.0085	0.0002	0.0096
Variance of Conventional Moments	25	0.0658	0.0089	0.0006	0.0278
	50	0.0570	0.0057	0.0002	0.0239
	100	0.0343	0.0045	0.0001	0.0099
Variance of Alternate Moments	25	0.0782	0.0263	0.0010	0.0428
	50	0.0519	0.0127	0.0002	0.0294
	100	0.0208	0.0108	0.0002	0.0127

Table 6.6 Lambda Parameter Estimates of “Gamma” Distributed Samples

Table 6.7 lists the lambda parameter estimations of the “Exponential” distributed data samples from the Monte Carlo simulation study. In this case, one cannot conclude that any method is substantially more accurate than the others. The average estimates closest to the

theoretical values are spread evenly over the different methods and sample sizes. However, these results indicate the alternate moments may be the most consistent estimators. The method of alternate moments has the lowest variance values of the three methods.

"Exponential" Distribution					
Fitment Method	Sample Size	λ_1	λ_2	λ_3	λ_4
Theoretical	∞	0	-.00163	-.000009	-.00162
Average of Linear Moments	25	0.0386	-.0794	-.0064	-.0187
	50	0.0099	0.0277	-.0013	0.0515
	100	-.0087	0.0170	-.0002	0.0272
Average of Conventional Moments	25	0.1179	0.1522	0.1756	0.1900
	50	0.1378	0.0911	-.0001	0.1309
	100	0.1486	0.0558	-.0001	0.0722
Average of Alternate Moments	25	-.0185	0.0502	-.0012	0.0905
	50	-.0006	0.0560	0.0000072	0.0711
	100	-.0196	0.0426	0.000071	0.0501
Variance of Linear Moments	25	0.0345	0.0624	0.0004	0.0344
	50	0.0168	0.0203	0.00004	0.0201
	100	0.0041	0.0100	0.000008	0.0085
Variance of Conventional Moments	25	0.2459	0.0376	0.4469	0.0403
	50	0.1000	0.0113	.00000003	0.0257
	100	0.0654	0.0080	.00000002	0.0128
Variance of Alternate Moments	25	0.0246	0.0354	0.000084	0.0365
	50	0.0059	0.0126	0.0000037	0.0151
	100	0.0024	0.0060	0.0000011	0.0066

Table 6.7 Lambda Parameter Estimates of "Exponential" Distributed Samples

As noted for the previous distributions, the variance decreases as the sample size increases, for all three methods. The lambda parameter values for the theoretical "Gamma" distribution are about three times smaller than the parameter values of the theoretical "Normal" distribution, and the parameter values of the theoretical "Exponential" distribution are about twenty times smaller than the parameters of the "Gamma" distribution. Thus, a given change in a lambda parameter is more likely to significantly alter the appearance of an estimated "Exponential" fit than an estimated "Normal" fit, even though the change may not affect the K-S values for goodness–of–fit to the same degree. This can account for the discrepancies noted between the visual PDF's and the Kolmogorov–Smirnov statistics from the comparable CDF's. The PDF's showed that the number of poor fits increased from "Normal" to "Gamma" to "Exponential" shapes. However, the averages of the K-S statistics decreased along the same line of comparison. Even though the K-S statistics were small, the difference in lambda parameters needed to change a "good fit" to a "poor fit" is even smaller.

6.3 Fitted Distributions vs. Empirical Distributions

Until now, the results have examined the three methods' abilities to closely fit a distribution to the underlying theoretical distribution. The methods estimate a distribution via sample data, rather than directly from the theoretical distribution. This intermediate step is the source of the "sampling variability," or "noise." We can now also examine how well each of the methods fit a distribution to the empirical CDF of the individual data samples. This ability is often of interest to a modeler, who is concerned with closely fitting a distribution to sample data when the underlying distribution of the population is unknown and is used as a barometer for assessing goodness–of–fit.

6.3.1 Summarizing Kolmogorov–Smirnov Statistics. Table 6.8 summarizes Kolmogorov–Smirnov (K-S) statistics collected from each of the 27 Monte Carlo experiments. The "Average K-S" Column lists the average over the 30 samples of the average difference between the

fitted and empirical distribution functions. The "Maximum K-S" Column lists the average over the 30 samples of the maximum difference between the same functions. The "Max K-S Variance" Column lists the variance of the 30 values which form the average in the previous column.

Unlike the K-S statistics collected from the theoretical distributions, these statistics indicate small but consistent advantages for the linear moments and the alternate moments over the conventional moments. Regarding the variances of the maximum K-S values, no method shows a clear advantage for "Normal" distributed samples, but both the linear and alternate methods average only half as much variance as the conventional moments for "Gamma" and "Exponential" distributed sample data. With both consistently lower means and variances of the K-S values, the methods of linear and alternate moments are superior to conventional moments for estimating and fitting distributions to sample data.

Distribution Shape	Sample Size	Method of Moments	Average K-S	Maximum K-S	Max K-S Variance
"Normal"	25	linear	0.02055	0.09573	.00052
		conventional	0.02146	0.10001	.00055
		alternate	0.02042	0.09788	.00047
	50	linear	0.01438	0.06771	.00031
		conventional	0.01576	0.07544	.00061
		alternate	0.01437	0.07069	.00036
	100	linear	0.01001	0.04640	.00014
		conventional	0.01142	0.05237	.00025
		alternate	0.01039	0.05005	.00024
"Gamma"	25	linear	0.01720	0.09493	.00063
		conventional	0.01899	0.11345	.00096
		alternate	0.01731	0.10338	.00074
	50	linear	0.01198	0.06632	.00030
		conventional	0.01552	0.08983	.00079
		alternate	0.01200	0.06965	.00034
	100	linear	0.00855	0.04647	.00019
		conventional	0.01272	0.07009	.00040
		alternate	0.00871	0.04963	.00024
"Exponential"	25	linear	0.01841	0.09955	.00074
		conventional	0.01983	0.13500	.00138
		alternate	0.01767	0.11254	.00130
	50	linear	0.01196	0.07062	.00035
		conventional	0.01524	0.10608	.00110
		alternate	0.01239	0.07479	.00035
	100	linear	0.00855	0.04811	.00021
		conventional	0.01257	0.08529	.00083
		alternate	0.00907	0.05183	.00040

Table 6.8 K-S Statistic Summary of fitted CDF's vs. empirical CDF's

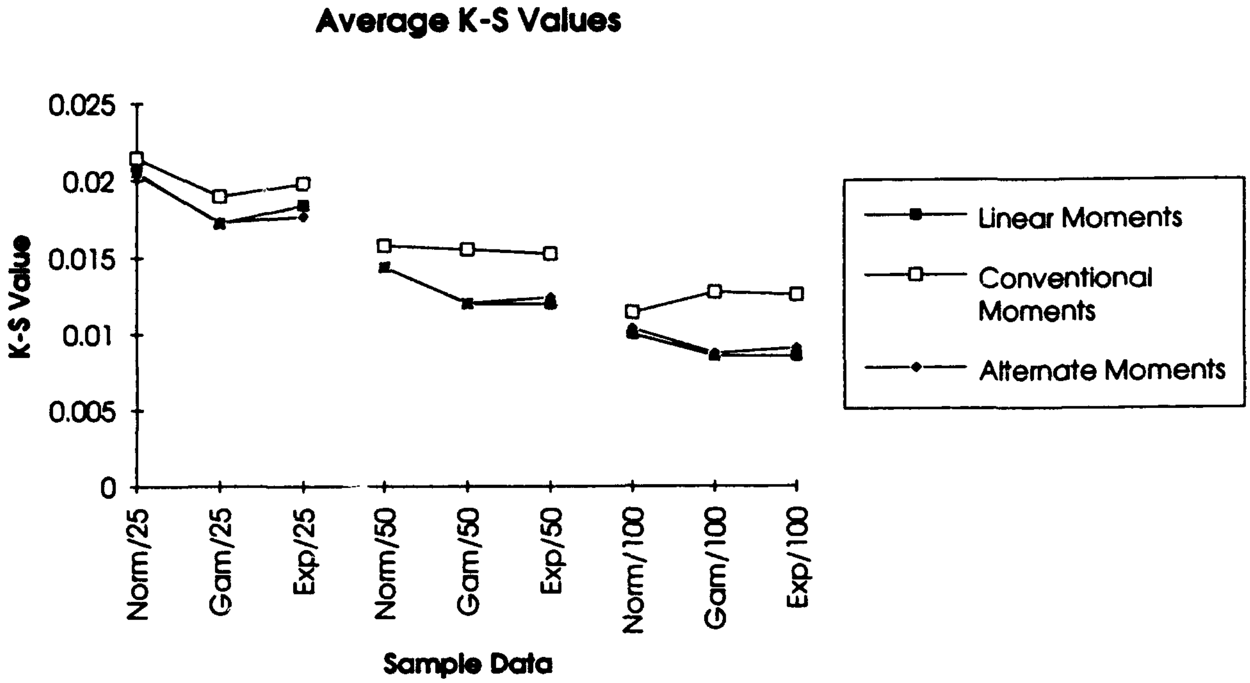

Figure 6.16 Average K-S Values of fitted CDF's vs. empirical CDF's

Figures 6.18 through 6.20 display the information from the "Average K-S," "Maximum K-S," and "Max K-S Variance" columns of Table 6.8, respectively.

Figure 6.18 clearly shows that the linear and alternate moments are comparable while the conventional moments produce slightly higher averages of the K-S values. There are two downward trends depicted in the chart. The average K-S values decrease as the sample size increases, and as the skewness and kurtosis of the theoretical distribution increases.

Figure 6.19 shows even more disparity between the conventional moments and the other two methods. Note that for all cases of "Normal" distributed sample data, all three methods perform equally well. For "Gamma" and "Exponential" distributed sample data, the K-S values associated with teh method of conventional moments become incresingly larger than those for the other methods as the skewness and kurtosis increases. Also, for the first time, linear moments show a small but consistent improvement over alternate moments. Figure

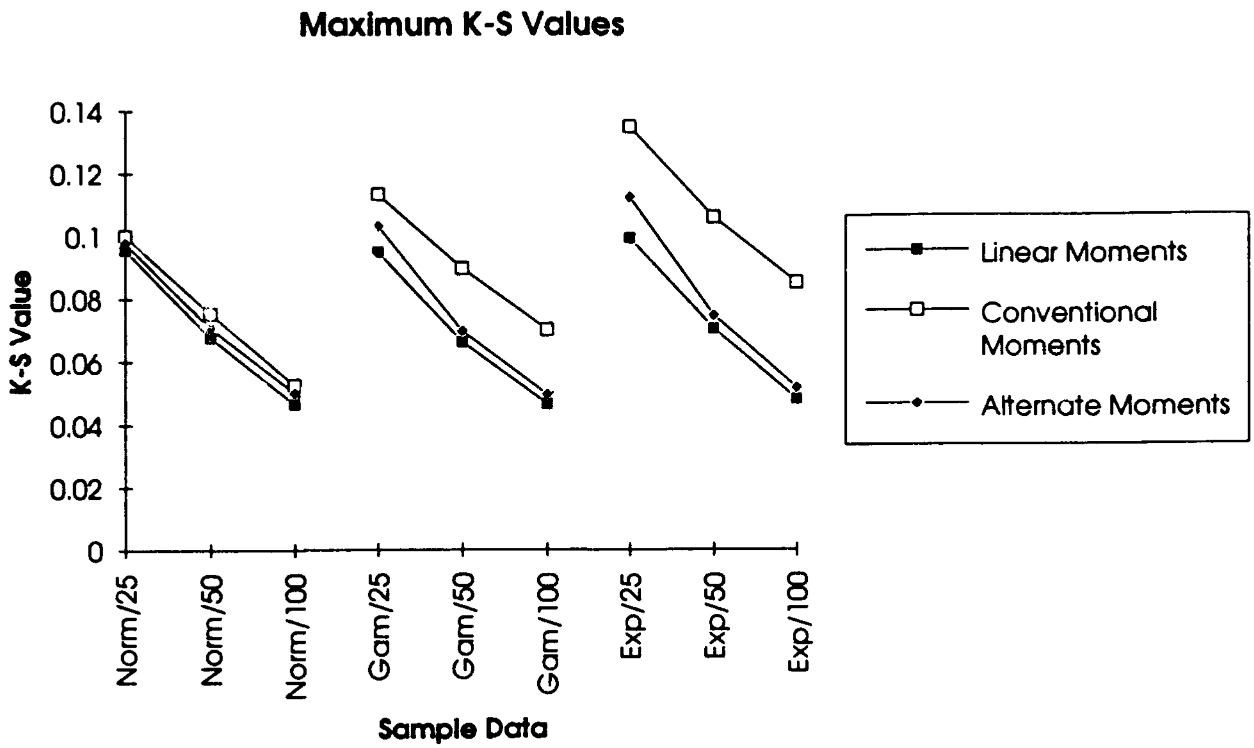

Figure 6.17 Maximum K-S Values of fitted CDF's vs. empirical CDF's

6.19 also shows a reversal of a previous trend: the K-S values increase from the "Normal" to the "Gamma," and from the "Gamma" to the "Exponential" sample data, for a constant sample size.

Figure 6.20 indicates that all three methods perform equally well for all cases of the "Normal" distribution. Alternate moments are less consistent than linear moments for 25-element samples but are equally consistent for larger samples. Conventional moments show increasing variance for "Gamma" and "Exponential" distributions. The K-S values increase with increasing values of skewness and kurtosis.

The increasing differences in the K-S values also suggest that the method of conventional moments becomes less suitable as the skewness and kurtosis of the distribution of sample data increase. There is little change in differences between K-S values as the sample size changes, as depicted by the nearly parallel lines on the chart. This suggests that, while

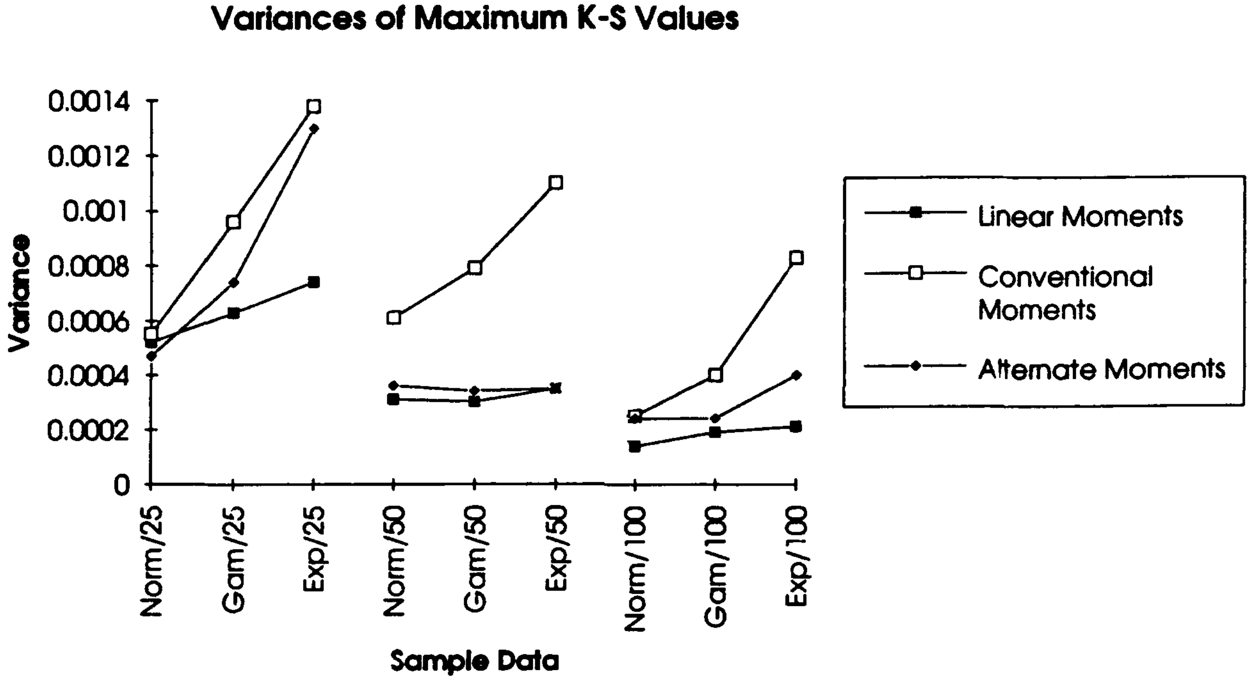

Figure 6.18 Variances of Maximum K-S Values of fitted CDF's vs. empirical CDF's

the methods of linear and alternate moments are better than the method of conventional moments overall, their relative superiority is less for small data samples. Also, this does not necessarily indicate an advantage in fitting distributions to the underlying distribution of the population.

Table 6.8 and Figures 6.18 through 6.20 provide a cursory overall look at the objective measures of the goodness-of-fit for each of the three methods of fit. The mean and variance of the K-S statistics provide a feel for which method will perform well in the long run. and will provide consistently-close fits. But, information on which method will most often provide the closest fit is also useful.

6.3.2 Kolmogorov-Smirnov Statistics of Individual Data Samples. Tables 6.9 and 6.10 list the number of times that each method of fit scored the lowest K-S value among each in a set of thirty samples. The tables are divided into summaries of maximum and average

K-S values. Each table is further broken down by the underlying distribution and size of the samples in the set.

Fitted Distribution vs. Empirical Distribution				
Distribution Shape	Sample Size	MAX K-S of Linear Method	MAX K-S of Conventional Method	MAX K-S of Alternate Method
"Normal"	25	14	8	8
	50	16	5	9
	100	20	4	6
"Gamma"	25	21	4	5
	50	20	2	8
	100	17	3	10
"Exponential"	25	18	5	7
	50	20	1	9
	100	14	2	14

Table 6.9 Number of Lowest MAX K-S Empirical Differences per set of 30 samples

Table 6.9 shows that the method of linear moments most often estimates the best fit to the distribution of the sample data for all distribution shapes. The linear moments show clear superiority over both of the other methods. However, no method shows consistent increasing or decreasing performance as the sample size changes independent of the sample distribution. Most notable is the clear advantage that the method of linear moments possesses in light of the lack of similar evidence in Table 6.3.

Fitted Distribution vs. Empirical Distribution				
Distribution Shape	Sample Size	AVG K-S of Linear Method	AVG K-S of Conventional Method	AVG K-S of Alternate Method
"Normal"	25	8	12	10
	50	8	11	11
	100	19	3	8
"Gamma"	25	12	7	11
	50	14	2	14
	100	14	2	14
"Exponential"	25	9	6	15
	50	14	6	10
	100	16	3	11

Table 6.10 Number of Lowest AVG K-S Empirical Differences per set of 30 samples

Table 6.10 shows that none of the methods are better estimators of the distribution of "Normal" distributed sample data than any other. However, it does show that both the method of linear moments and the method of alternate moments are clearly better estimators of empirical distributions than the method of conventional moments. Still, there is no change in performance which corresponds to changes in the sample size that is independent of the distribution shape.

Once the information was gleaned from the data collected from the Monte Carlo experiment, some insight was gained into the method of linear moments. These findings and the conclusions drawn from them are presented in Chapter Seven.

VII. Conclusions and Recommendations

7.1 The Generalized Lambda Distribution and Linear Moments

The purpose of thesis was to assess the usefulness of the method of L-moments applied to the GLD, especially in relation to other, more well–established methods for parameter estimation. The specific goal was to determine whether any one method is superior in fitting the generalized lambda distribution to sample data when the underlying distribution is unknown and the sample size is relatively small.

The results in Chapter Six suggest that the methods of linear and alternate moments perform about equally well and are both superior to the conventional method of moments for fitting the estimated GLD function to an underlying theoretical distribution. They also indicate that the methods of linear and alternate moments perform equally well. This superiority is evident in the low averages and variances of the Kolmogorov–Smirnov statistics, the accurate averages and low variances of the estimated lambda parameters, and the high percentages of properly–shaped PDF's generated from the sets of sample data. Thus, linear moments can be considered an improvement on the methods of conventional and alternate moments.

The results in Chapter Six also suggest that the method of linear moments is better than the other two methods tested for fitting the GLD to the empirical distributions of sample data. This superiority is evident in the low averages and variances of the Kolmogorov–Smirnov statistics collected from comparisons of the fitted distributions and the empirical distributions of sample data.

In addition, the method of linear moments showed the greatest flexibility in fitting the GLD to sample data from different distributions. The method of L-moments estimated the GLD from "Gamma" distributed sample data as well as it did from normal–distributed sample data, and there was only moderate degradation in the estimates of "Exponential" distributed samples. The method of alternate moments showed increasing difficulty with

both the "Gamma" and "Exponential" samples. The conventional method of moments had difficulty when dealing with "Gamma" distributed and "Exponential" distributed sample data. It appears that the asymmetry and convexity, as measured by the skewness and kurtosis, respectively, of sample data can also have a negative effect on the goodness-of-fit for some methods of estimation, such as the conventional method of moments. The results of this experiment show that linear moments and alternate moments are much less affected by the asymmetry and convexity of the empirical distributions of sample data.

In general, the method of linear moments shows an increasing advantage as the asymmetry and convexity of the distribution of sample data increase, and a constant but small advantage over the other methods tested as the sample size decreases. However, these advantages are gained primarily when fitting estimated distributions to the empirical distribution of an individual sample. The advantage was less evident when using linear moments to estimate the underlying theoretical distribution from sample data than when using the other two methods.

I recommend that those involved in modeling and simulation use linear moments as the preferred method when fitting the Generalized Lambda Distribution to sample data. In addition to my findings and conclusions, I have recommendations for further research in the following areas.

7.2 GLD Software Package, Version II, in C++

Hsu (1991) developed a complete software package using the GLD and various estimating methods to fit distributions to sample data and to derive lambda parameters from various known statistics. The software package is written in $C++$, a state-of-the-art, object-oriented computer language. The program is menu-driven, user-interactive, and employs real-time color graphics. The underlying algorithms of the program employ the conventional method of moments and the alternate method of moments, as well as others. L-moments should be incorporated into a new version of the software to enhance its capabilities.

7.3 Using Powell's Algorithm with Hosking's Lambda Distribution

As discussed in Chapter Four, Powell's Algorithm is the numerical search routine of choice for this study and for the GLD. However, it is still not completely proven to be reliable when searching back and forth between the positive and negative regions. Hosking (August 1993) encountered the same problem using Newton's method and, instead of revising the search routine as Mykytka has done, he revised the lambda distribution quantile function. Hosking's Lambda distribution could be used with Powell's Algorithm to create a more reliable, faster search routine. This combination would permit the search routine to begin at the origin, proceed with the search in the right direction the first time, and revisit the origin if needed, avoiding the forbidden zones during transitions between regions.

7.4 Expanding the GLD's Range of Approximate Distribution Shapes

As Hosking noted, the sample L-moments exist if the sample has a finite mean. This does not always hold true for conventional moments. Thus, it is possible that the method of linear moments will permit modelers to compute statistics from samples that previously had no existing conventional moments. Given the sample L-moments, GLD parameters could be calculated, and a distribution could be fit. These fitted distributions may extend into the regions of skewness and kurtosis previously unusable by the GLD, as depicted in Figure 2.2.

Appendix A. Comparisons of L-Moment Equations

In this Appendix, I present an in-depth comparison of the similarities and differences of the equations for the four L-moments, $\Lambda_1, \Lambda_2, \Lambda_3$, and Λ_4 from the four parameters the Generalized Lambda Distribution and the five parameters of Hosking's Lambda Distribution.

I begin by comparing the quantile function of Hosking's Lambda Distribution to the percentile function of the Generalized Lambda Distribution to determine the conversion factors between the two functions.

Starting with Hosking's quantile function (Hosking, 1986: 83):

$$x(F) = \xi + \alpha F^{\beta} - \gamma(1-F)^{\delta}, \quad 0 \leq F \leq 1. \tag{A.1}$$

By setting $\alpha = \gamma$, Hosking converts this five parameter function to a four parameter function (Hosking, 1993: 1):

$$x(F) = \xi + \alpha F^{\beta} - \alpha(1-F)^{\delta}, \quad 0 \leq F \leq 1. \tag{A.2}$$

I now introduce a new term, ϵ, which equals the inverse of Hosking's α term:

$$\alpha = \frac{1}{\epsilon}, \tag{A.3}$$

$$x(F) = \xi + \frac{F^{\beta}}{\epsilon} - \frac{(1-F)^{\delta}}{\epsilon}, \quad 0 \leq F \leq 1. \tag{A.4}$$

Combining the two terms into a single term with a common denominator:

$$x(F) = \xi + \frac{F^{\beta} - (1-F)^{\delta}}{\epsilon}, \quad 0 \le F \le 1. \tag{A.5}$$

Replacing Hosking's variable names with terminology used by Ramberg for the GLD, $F = p, x(F) = R(p), \xi = \lambda_1, \epsilon = \lambda_2, \beta = \lambda_3, \delta = \lambda_4$, results in the percentile function of the GLD (Bergevin, 1993: 7):

$$R(p) = \lambda_1 + \frac{p^{\lambda_3} - (1-p)^{\lambda_4}}{\lambda_2}, \quad 0 \le p \le 1. \tag{A.6}$$

Therefore, the two operations necessary to convert Hosking's Lambda Distribution to Ramberg's Generalized Lambda Distribution are to set the second and fourth parameters, α and γ equal to each other; and then to replace them with their inverse, ϵ. These two operations will be used to show the similarity between L-moment equations of Hosking's Lambda Distribution and the GLD.

Beginning with Hosking's equation for the first L-moment (Hosking, 1986: 83):

$$\Lambda_1 = \xi + \frac{\alpha}{1+\beta} - \frac{\gamma}{1+\delta}. \tag{A.7}$$

Cross-multiplying for a common denominator:

$$\Lambda_1 = \xi + \frac{\alpha(1+\delta)}{(1+\beta)(1+\delta)} - \frac{\gamma(1+\beta)}{(1+\delta)(1+\beta)}. \tag{A.8}$$

Combining the terms with a common denominator:

$$\Lambda_1 = \xi + \frac{\alpha(1+\delta) - \gamma(1+\beta)}{(1+\delta)(1+\beta)}. \tag{A.9}$$

Replacing α and γ with ϵ:

$$\Lambda_1 = \xi + \frac{(1+\delta)-(1+\beta)}{\epsilon(1+\delta)(1+\beta)}. \tag{A.10}$$

Simplifying the numerator:

$$\Lambda_1 = \xi + \frac{\delta-\beta}{\epsilon(1+\delta)(1+\beta)}. \tag{A.11}$$

Replacing Hosking's terms with Ramberg's terms yields the equation for Λ_1 derived by Bergevin (Bergevin, 1993: 64):

$$\Lambda_1 = \lambda_1 + \frac{\lambda_4-\lambda_3}{\lambda_2(\lambda_3+1)(\lambda_4+1)}. \tag{A.12}$$

Performing the same operations on Hosking's equation for the second L-moment (Hosking, 1986: 83):

$$\Lambda_2 = \frac{\alpha\beta}{(1+\beta)(2+\beta)} + \frac{\gamma\delta}{(1+\delta)(2+\delta)}. \tag{A.13}$$

Cross-multiplying for a common denominator:

$$\Lambda_2 = \frac{\alpha\beta(1+\delta)(2+\delta)}{(1+\beta)(2+\beta)(1+\delta)(2+\delta)} + \frac{\gamma\delta(1+\beta)(2+\beta)}{(1+\delta)(2+\delta)(1+\beta)(2+\beta)}. \tag{A.14}$$

Combining the terms with a common denominator:

$$\Lambda_2 = \frac{\alpha\beta(1+\delta)(2+\delta)+\gamma\delta(1+\beta)(2+\beta)}{(1+\delta)(2+\delta)(1+\beta)(2+\beta)}. \tag{A.15}$$

Replacing α and γ with ϵ:

$$\Lambda_2 = \frac{\beta(1+\delta)(2+\delta)+\delta(1+\beta)(2+\beta)}{\epsilon(1+\delta)(2+\delta)(1+\beta)(2+\beta)}. \tag{A.16}$$

Replacing Hosking's terms with Ramberg's terms yields the equation for Λ_2 derived by Bergevin (Bergevin, 1993: 65):

$$\Lambda_2 = \frac{\lambda_3(\lambda_4+1)(\lambda_4+2)+\lambda_4(\lambda_3+1)(\lambda_3+2)}{\lambda_2(\lambda_3+1)(\lambda_3+2)(\lambda_4+1)(\lambda_4+2)}. \tag{A.17}$$

Performing these operations on Hosking's equation for the third L-moment (Hosking, 1986: 83):

$$\Lambda_3 = \frac{\alpha\beta(\beta-1)}{(1+\beta)(2+\beta)(3+\beta)} - \frac{\gamma\delta(\delta-1)}{(1+\delta)(2+\delta)(3+\delta)}. \tag{A.18}$$

Cross-multiplying for a common denominator:

$$\Lambda_3 = \frac{\alpha\beta(\beta-1)(1+\delta)(2+\delta)(3+\delta)}{(1+\beta)(2+\beta)(3+\beta)(1+\delta)(2+\delta)(3+\delta)} - \frac{\gamma\delta(\delta-1)(1+\beta)(2+\beta)(3+\beta)}{(1+\delta)(2+\delta)(3+\delta)(1+\beta)(2+\beta)(3+\beta)}. \tag{A.19}$$

Combining the terms with a common denominator:

$$\Lambda_3 = \frac{\alpha\beta(\beta-1)(1+\delta)(2+\delta)(3+\delta)-\gamma\delta(\delta-1)(1+\beta)(2+\beta)(3+\beta)}{(1+\delta)(2+\delta)(3+\delta)(1+\beta)(2+\beta)(3+\beta)}. \tag{A.20}$$

Replacing α and γ with ϵ:

$$\Lambda_3 = \frac{\beta(\beta-1)(1+\delta)(2+\delta)(3+\delta)-\delta(\delta-1)(1+\beta)(2+\beta)(3+\beta)}{\epsilon(1+\delta)(2+\delta)(3+\delta)(1+\beta)(2+\beta)(3+\beta)}. \tag{A.21}$$

Simplifying the numerator:

$$\Lambda_3 = \frac{(\beta^2-\beta)(1+\delta)(2+\delta)(3+\delta)-(\delta^2-\delta)(1+\beta)(2+\beta)(3+\beta)}{\epsilon(1+\delta)(2+\delta)(3+\delta)(1+\beta)(2+\beta)(3+\beta)}. \tag{A.22}$$

Replacing Hosking's terms with Ramberg's terms yields the equation for Λ_3 derived by Bergevin (Bergevin, 1993: 66):

$$\Lambda_3 = \frac{(\lambda_3^2-\lambda_3)(\lambda_4+1)(\lambda_4+2)(\lambda_4+3)-(\lambda_4^2-\lambda_4)(\lambda_3+1)(\lambda_3+2)(\lambda_3+3)}{\lambda_2(\lambda_3+1)(\lambda_3+2)(\lambda_3+3)(\lambda_4+1)(\lambda_4+2)(\lambda_4+3)}. \tag{A.23}$$

Finally, performing these operations on Hosking's equation for the fourth L-moment (Hosking, 1986: 83):

$$\Lambda_4 = \frac{\alpha\beta(\beta-1)(\beta-2)}{(1+\beta)(2+\beta)(3+\beta)(4+\beta)} + \frac{\gamma\delta(\delta-1)(\delta-2)}{(1+\delta)(2+\delta)(3+\delta)(4+\delta)}. \tag{A.24}$$

Cross-multiplying for a common denominator:

$$\begin{aligned}\Lambda_4 = {} & \frac{\alpha\beta(\beta-1)(\beta-2)(1+\delta)(2+\delta)(3+\delta)(4+\delta)}{(1+\beta)(2+\beta)(3+\beta)(4+\beta)(1+\delta)(2+\delta)(3+\delta)(4+\delta)} \\ & + \frac{\gamma\delta(\delta-1)(\delta-2)(1+\beta)(2+\beta)(3+\beta)(4+\beta)}{(1+\delta)(2+\delta)(3+\delta)(4+\delta)(1+\beta)(2+\beta)(3+\beta)(4+\beta)}.\end{aligned} \tag{A.25}$$

Combining the terms with a common denominator:

$$\Lambda_4 = \frac{\alpha\beta(\beta-1)(\beta-2)(1+\delta)(2+\delta)(3+\delta)(4+\delta)+\gamma\delta(\delta-1)(\delta-2)(1+\beta)(2+\beta)(3+\beta)(4+\beta)}{(1+\delta)(2+\delta)(3+\delta)(4+\delta)(1+\beta)(2+\beta)(3+\beta)(4+\beta)}. \tag{A.26}$$

Replacing α and γ with ϵ:

$$\Lambda_4 = \frac{\beta(\beta-1)(\beta-2)(1+\delta)(2+\delta)(3+\delta)(4+\delta)+\delta(\delta-1)(\delta-2)(1+\beta)(2+\beta)(3+\beta)(4+\beta)}{\epsilon(1+\delta)(2+\delta)(3+\delta)(4+\delta)(1+\beta)(2+\beta)(3+\beta)(4+\beta)}. \tag{A.27}$$

Simplifying the numerator:

$$\Lambda_4 = \frac{(\beta^3-3\beta^2+2\beta)(1+\delta)(2+\delta)(3+\delta)(4+\delta)+(\delta^3-3\delta^2-2\delta)(1+\beta)(2+\beta)(3+\beta)(4+\beta)}{\epsilon(1+\delta)(2+\delta)(3+\delta)(4+\delta)(1+\beta)(2+\beta)(3+\beta)(4+\beta)}. \tag{A.28}$$

Replacing Hosking's terms with Ramberg's terms yields the equation for Λ_4 derived by Bergevin (Bergevin, 1993: 69):

$$\Lambda_4 = \frac{(\lambda_3^3-3\lambda_3^2+2\lambda_3)(\lambda_4+1)(\lambda_4+2)(\lambda_4+3)(\lambda_4+4)+(\lambda_4^3-3\lambda_4^2+2\lambda_4)(\lambda_3+1)(\lambda_3+2)(\lambda_3+3)(\lambda_3+4)}{\lambda_2(\lambda_3+1)(\lambda_3+2)(\lambda_3+3)(\lambda_3+4)(\lambda_4+1)(\lambda_4+2)(\lambda_4+3)(\lambda_4+4)}. \tag{A.29}$$

Appendix B. Experiment Results: Normal Distribution, Linear Moments

Monte Carlo Experiment Results

Statistical Summary

Experiment Result		Statistic			
		Minimum	Maximum	Average	Variance
Moments of Sample Data	1st	-0.32920000	0.33291600	-0.08192267	0.03075791
	2nd	0.46495600	0.69995600	0.56678553	0.00435006
	3rd	-0.14980300	0.16846000	0.02440728	0.00632024
	4th	-0.01568870	0.20792600	0.09713558	0.00315052
Lambda Parameters	λ_1	-2.07855000	1.39302000	-0.28595313	0.73005042
	λ_2	-0.22071600	0.49963800	0.19608461	0.02915614
	λ_3	-0.08825540	1.00248000	0.18008392	0.06060880
	λ_4	-0.12274100	1.11750000	0.23446677	0.07488433
Theoretical K-S Statistics	MIN	0.00000211	0.00132966	0.00015583	0.00000008
	AVG	0.01000620	0.05861190	0.03572510	0.00016930
	MAX	0.02489270	0.18884700	0.10145870	0.00149412
Empirical K-S Statistics	MIN	0.00000000	0.00024583	0.00002845	0.00000000
	AVG	0.01290350	0.02903130	0.02055428	0.00002040
	MAX	0.06148730	0.14373100	0.09572504	0.00051787

Experiment Number: 1A

Samples: 30 samples of 25 elements each

Original function: lambda approximation of a normal function

Method of fitment: method of linear moments

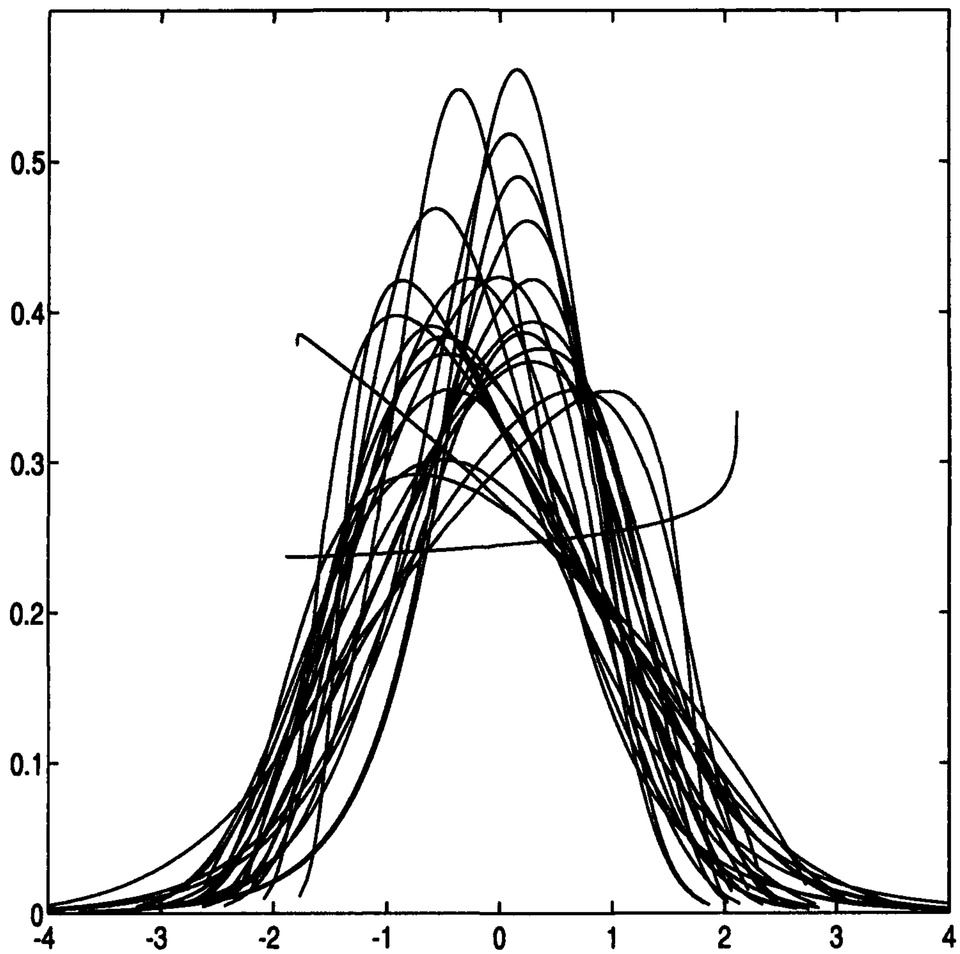

Figure B.1 Plots of 25-element samples from an approximate normal distribution fitted with linear moments

Monte Carlo Experiment Results

Statistical Summary

Experiment Result		Statistic: Minimum	Maximum	Average	Variance
Moments of Sample Data	1st	-0.24640400	0.34302800	-0.04351340	0.01692519
	2nd	0.46380700	0.65934200	0.56482997	0.00248374
	3rd	-0.12463400	0.08293850	-0.00737903	0.00265956
	4th	-0.00161649	0.18735300	0.10071303	0.00163888
Lambda Parameters	λ_1	-1.49791000	1.53929000	0.02692249	0.36133215
	λ_2	-0.12565200	0.60877900	0.23682851	0.01601138
	λ_3	-0.05730220	1.12361000	0.2[illegible]356539	0.05862897
	λ_4	-0.05661900	0.96255000	0.20318173	0.04691401
Theoretical K-S Statistics	MIN	0.00000482	0.00128289	0.00016679	0.00000010
	AVG	0.00678552	0.05570450	0.02451674	0.00010957
	MAX	0.02189700	0.13389000	0.06728391	0.00085707
Empirical K-S Statistics	MIN	0.00000000	0.00007557	0.00001988	0.00000000
	AVG	0.01005300	0.02212740	0.01435771	0.00001019
	MAX	0.04356000	0.12729100	0.06770739	0.00031097

Experiment Number: 1B

Samples: 30 samples of 50 elements each

Original function: lambda approximation of a normal function

Method of fitment: method of linear moments

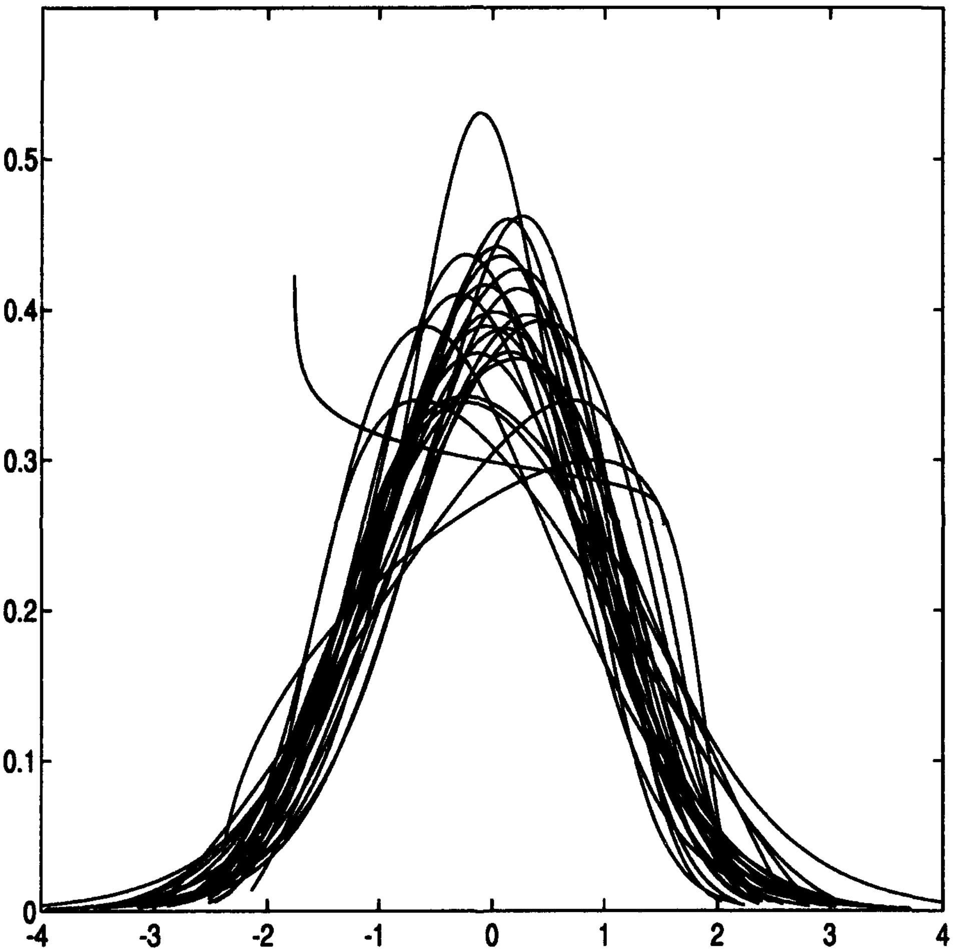

Figure B.2 Plots of 50-element samples from an approximate normal distribution fitted with linear moments

Monte Carlo Experiment Results

Statistical Summary

Experiment Result		Statistic			
		Minimum	Maximum	Average	Variance
Moments of Sample Data	1st	-0.21766800	0.23810900	-0.05008467	0.01089713
	2nd	0.50551300	0.61714400	0.55958817	0.00074421
	3rd	-0.09907760	0.08210260	-0.00073607	0.00193096
	4th	0.06930190	0.16791000	0.11116174	0.00041959
Lambda Parameters	λ_1	-0.65020500	0.74872200	-0.05474650	0.12705271
	λ_2	-0.00664537	0.38615300	0.23400949	0.00608533
	λ_3	-0.00288072	0.49463600	0.17515935	0.00994712
	λ_4	-0.00406039	0.40454100	0.18023218	0.00977001
Theoretical K-S Statistics	MIN	0.00000004	0.00123381	0.00021441	0.00000016
	AVG	0.00733220	0.03906030	0.01885646	0.00007125
	MAX	0.01486220	0.08808770	0.05100268	0.00051479
Empirical K-S Statistics	MIN	0.00000000	0.00011953	0.00002576	0.00000000
	AVG	0.00657308	0.01709660	0.01000614	0.00000579
	MAX	0.02922960	0.08260010	0.04640222	0.00014395

Experiment Number: 1C

Samples: 30 samples of 100 elements each

Original function: lambda approximation of a normal function

Method of fitment: method of linear moments

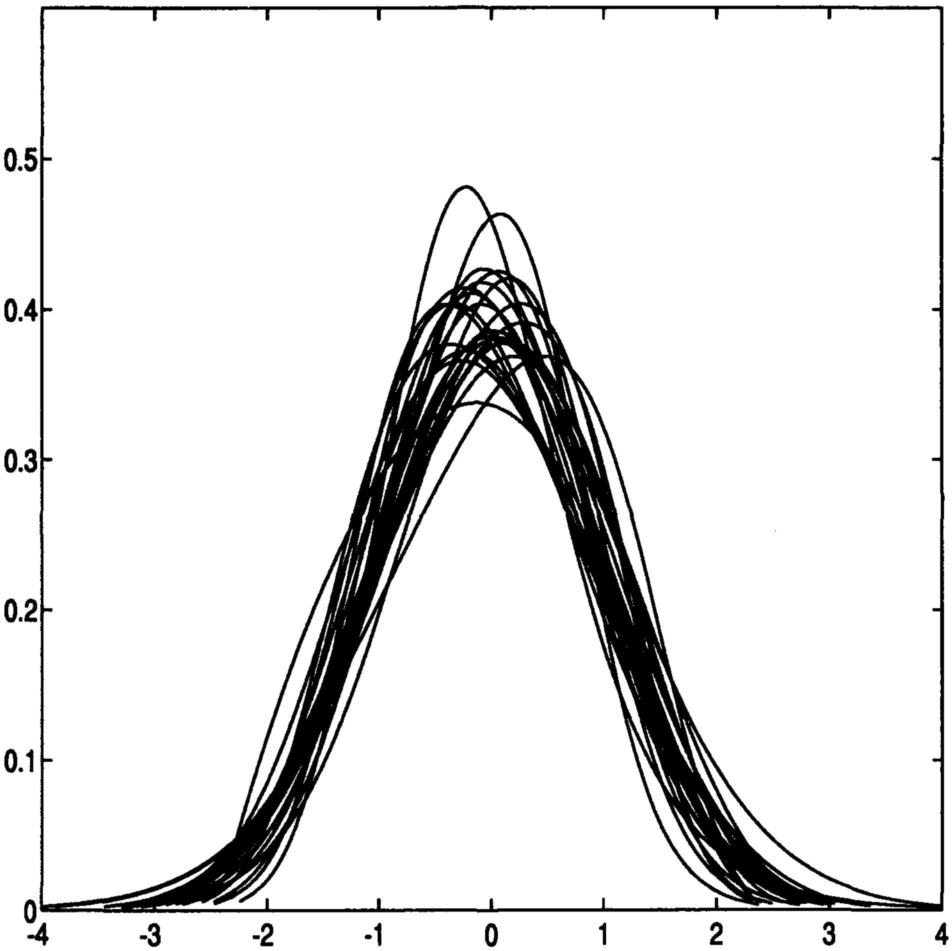

Figure B.3 Plots of 100-element samples from an approximate normal distribution fitted with linear moments

Appendix C. Experiment Results: Gamma Distribution, Linear Moments

Monte Carlo Experiment Results

Statistical Summary

Experiment Result		Statistic			
		Minimum	Maximum	Average	Variance
Moments of Sample Data	1st	0.48061600	1.17188000	0.74208173	0.03016900
	2nd	0.38186000	0.67193700	0.51891087	0.00718194
	3rd	0.07615490	0.37085800	0.23640660	0.00690586
	4th	0.02031400	0.29404600	0.14745152	0.00407120
Lambda Parameters	λ_1	-0.62057600	0.49394800	-0.12984145	0.09483345
	λ_2	-0.65561600	0.31523900	0.02458554	0.05222803
	λ_3	-0.10849900	0.05369720	-0.00112102	0.00122820
	λ_4	-0.32500000	0.51999700	0.08401853	0.03978563
Theoretical K-S Statistics	MIN	0.00000067	0.00088185	0.00011831	0.00000004
	AVG	0.01115250	0.04637010	0.02790449	0.00010405
	MAX	0.02728510	0.16391300	0.09577868	0.00132345
Empirical K-S Statistics	MIN	0.00000000	0.00022676	0.00002922	0.00000000
	AVG	0.01051120	0.02525230	0.01720370	0.00001793
	MAX	0.05963730	0.15913600	0.09493287	0.00063476

Experiment Number: 2A

Samples: 30 samples of 25 elements each

Original function: lambda approximation of a gamma function

Method of fitment: method of linear moments

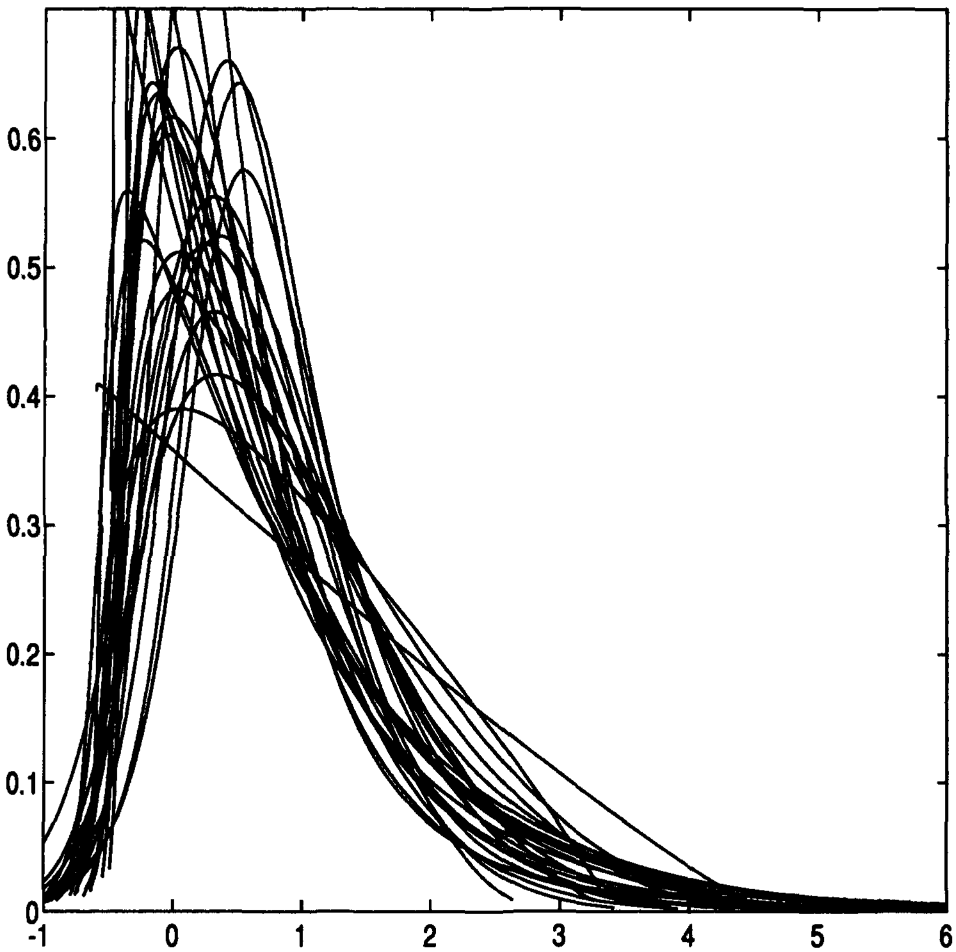

Figure C.1 Plots of 25-element samples from an approximate gamma distribution fitted with linear moments

Monte Carlo Experiment Results

Statistical Summary

Experiment Result		Statistic			
		Minimum	Maximum	Average	Variance
Moments of Sample Data	1st	0.55609200	1.23099000	0.76728553	0.01912877
	2nd	0.40011800	0.73080100	0.51610747	0.00496379
	3rd	0.10441400	0.31780000	0.20907797	0.00304550
	4th	0.04483840	0.22731200	0.13154492	0.00207545
Lambda Parameters	λ_1	-0.59335000	0.36391500	-0.10537867	0.06585115
	λ_2	-0.35408300	0.30237500	0.08758405	0.01873872
	λ_3	-0.06458690	0.04024890	0.00584606	0.00037417
	λ_4	-0.17902700	0.49252300	0.13232465	0.02834047
Theoretical K-S Statistics	MIN	0.00000033	0.00088185	0.00015311	0.00000006
	AVG	0.00420067	0.05758760	0.02040218	0.00009339
	MAX	0.02140450	0.12633600	0.06529886	0.00076624
Empirical K-S Statistics	MIN	0.00000000	0.00011152	0.00001303	0.00000000
	AVG	0.00687171	0.01760340	0.01197734	0.00000882
	MAX	0.04019940	0.11113000	0.06632050	0.00030197

Experiment Number: 2B

Samples: 30 samples of 50 elements each

Original function: lambda approximation of a gamma function

Method of fitment: method of linear moments

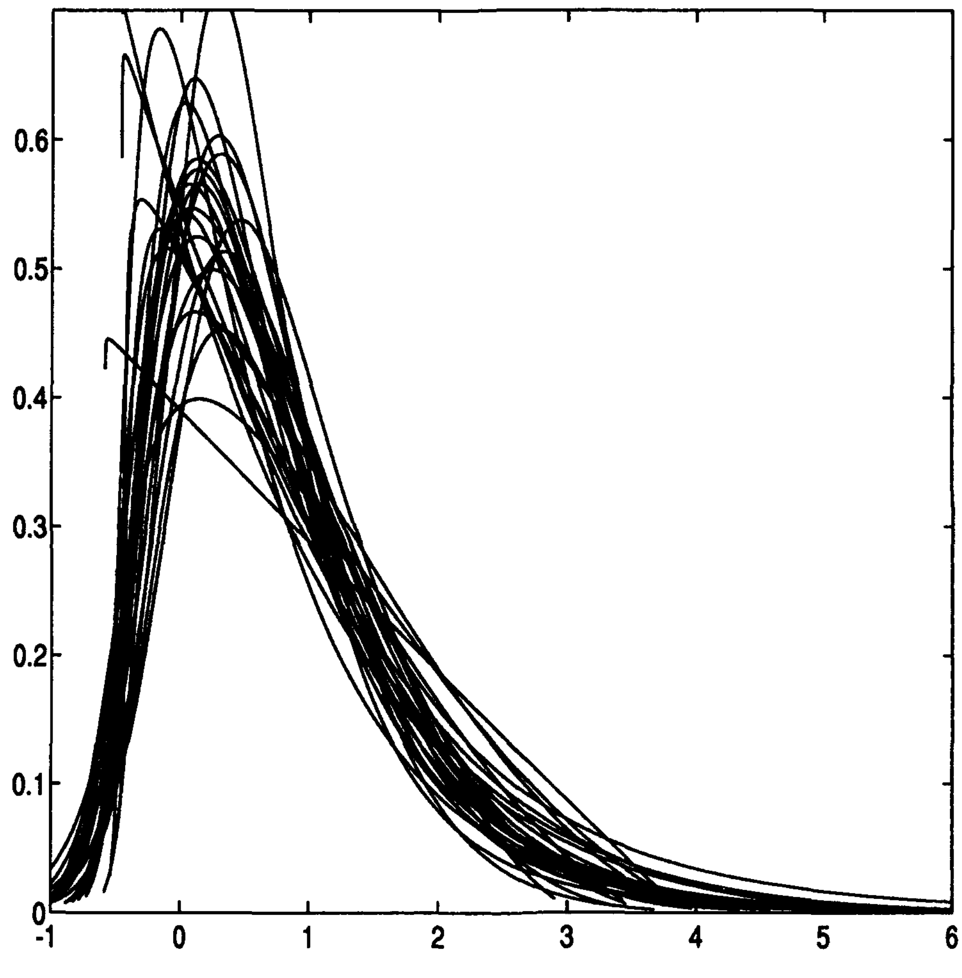

Figure C.2 Plots of 50-element samples from an approximate gamma distribution fitted with linear moments

Monte Carlo Experiment Results

Statistical Summary

Experiment Result		Statistic			
		Minimum	Maximum	Average	Variance
Moments of Sample Data	1st	0.57285600	1.09224000	0.76092617	0.01117079
	2nd	0.41755600	0.66484400	0.51482683	0.00213806
	3rd	0.12744100	0.29910900	0.22375943	0.00187228
	4th	0.07923440	0.22313900	0.14053823	0.00079948
Lambda Parameters	λ_1	-0.47898200	0.17442600	-0.08526832	0.02220021
	λ_2	-0.25092900	0.21989700	0.07842475	0.00847902
	λ_3	-0.03406480	0.03163950	0.00830060	0.00017585
	λ_4	-0.16210500	0.28390000	0.09417172	0.00964847
Theoretical K-S Statistics	MIN	0.00000111	0.00088185	0.00014947	0.00000006
	AVG	0.00632982	0.03989130	0.01549239	0.00005874
	MAX	0.01531930	0.08619750	0.05020063	0.00043966
Empirical K-S Statistics	MIN	0.00000000	0.00016237	0.00002068	0.00000000
	AVG	0.00517[illegible]03	0.01353660	0.00855233	0.00000408
	MAX	0.02793380	0.08175310	0.04647265	0.00018876

Experiment Number: 2C

Samples: 30 samples of 100 elements each

Original function: lambda approximation of a gamma function

Method of fitment: method of linear moments

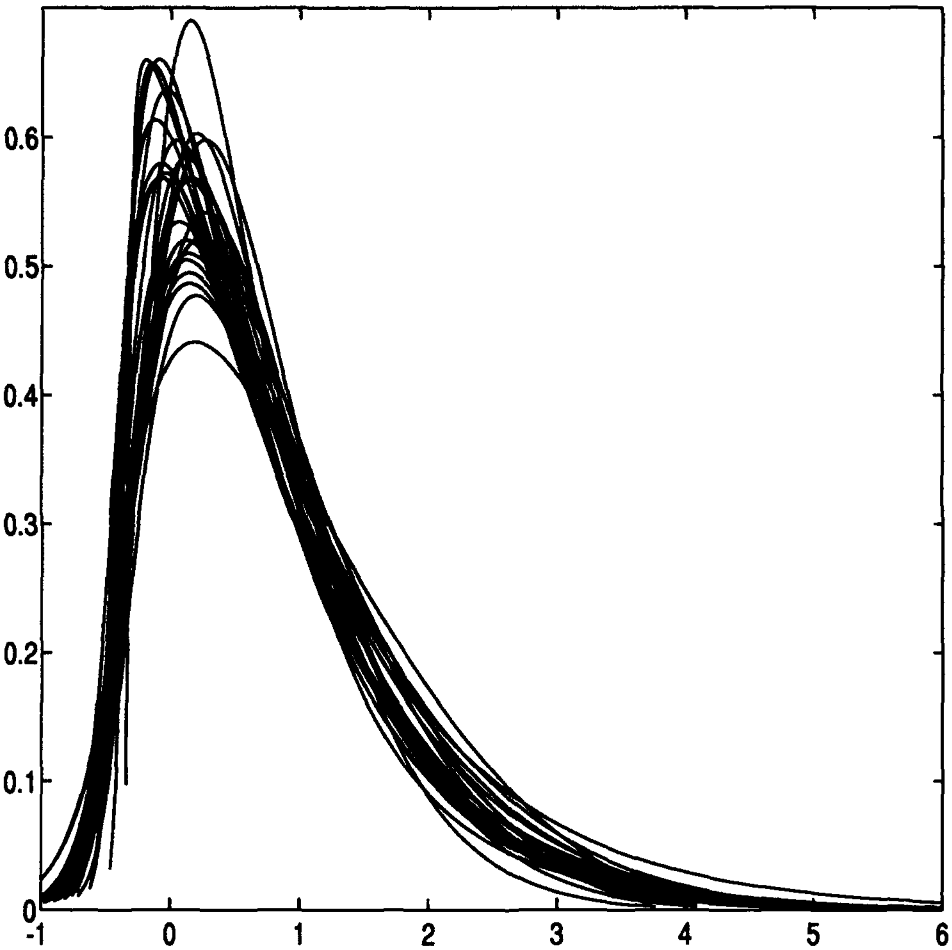

Figure C.3 Plots of 100-element samples from an approximate gamma distribution fitted with linear moments

Appendix D. Experiment Results: Exponential Distribution, Linear Moments

Monte Carlo Experiment Results

Statistical Summary

Experiment Result		Statistic			
		Minimum	Maximum	Average	Variance
Moments of Sample Data	1st	0.67087600	1.36818000	0.93589840	0.02967292
	2nd	0.32504700	0.68764400	0.48435410	0.00919645
	3rd	0.19025100	0.46263500	0.33083303	0.00652701
	4th	0.02826610	0.32878800	0.16516392	0.00527456
Lambda Parameters	λ_1	-0.24390200	0.54126400	0.03855568	0.03454463
	λ_2	-0.78781700	0.22410700	-0.07935519	0.06237191
	λ_3	-0.08832810	0.01016820	-0.00644237	0.00040783
	λ_4	-0.37972500	0.36723100	-0.01870103	0.03442562
Theoretical K-S Statistics	MIN	0.00000000	0.00127325	0.00022416	0.00000012
	AVG	0.00905325	0.04554390	0.02479719	0.00009789
	MAX	0.03506210	0.16055200	0.09236931	0.00127595
Empirical K-S Statistics	MIN	0.00000000	0.00021511	0.00006333	0.00000000
	AVG	0.00968736	0.04757100	0.01840903	0.00006827
	MAX	0.05938320	0.18146000	0.09954951	0.00074020

Experiment Number: 3A

Samples: 30 samples of 25 elements each

Original function: lambda approximation of an exponential function

Method of fitment: method of linear moments

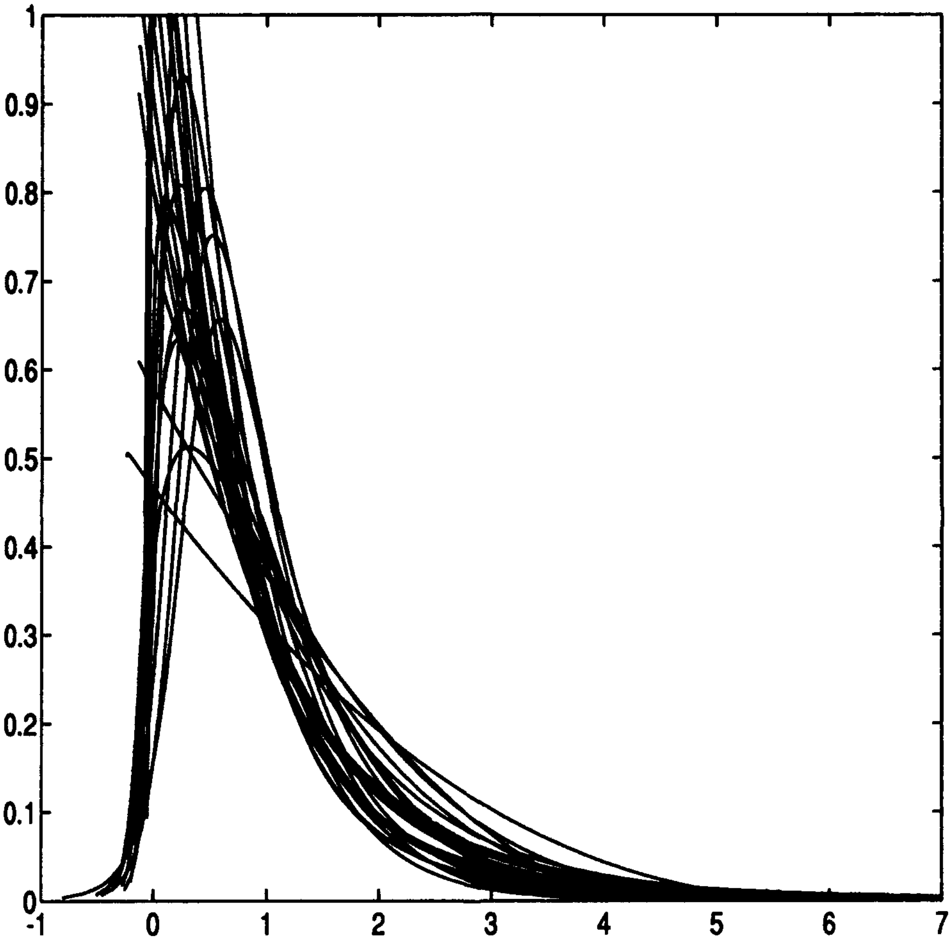

Figure D.1 Plots of 25-element samples from an approximate exponential distribution fitted with linear moments

Monte Carlo Experiment Results

Statistical Summary

Experiment Result		Statistic			
		Minimum	Maximum	Average	Variance
Moments of Sample Data	1st	0.75096200	1.44070000	0.95519720	0.01969888
	2nd	0.35598200	0.75015500	0.48163140	0.00651215
	3rd	0.22137500	0.42617500	0.30446793	0.00300091
	4th	0.04726550	0.24158300	0.14203869	0.00257635
Lambda Parameters	λ_1	-0.20071400	0.33470900	0.00991586	0.01683758
	λ_2	-0.39034500	0.26419000	0.02766978	0.02028058
	λ_3	-0.03202840	0.00766124	-0.00126660	0.00004343
	λ_4	-0.20218900	0.32630700	0.05152388	0.02006486
Theoretical K-S Statistics	MIN	0.00000074	0.00187586	0.00034786	0.00000026
	AVG	0.00366195	0.06010810	0.01895440	0.00010956
	MAX	0.02063870	0.11822400	0.06403366	0.00063196
Empirical K-S Statistics	MIN	0.00000000	0.00028062	0.00002655	0.00000000
	AVG	0.00600925	0.01978000	0.01196005	0.00001157
	MAX	0.03931390	0.10229500	0.07062143	0.00035156

Experiment Number: 3B

Samples: 30 samples of 50 elements each

Original function: lambda approximation of an exponential function

Method of fitment: method of linear moments

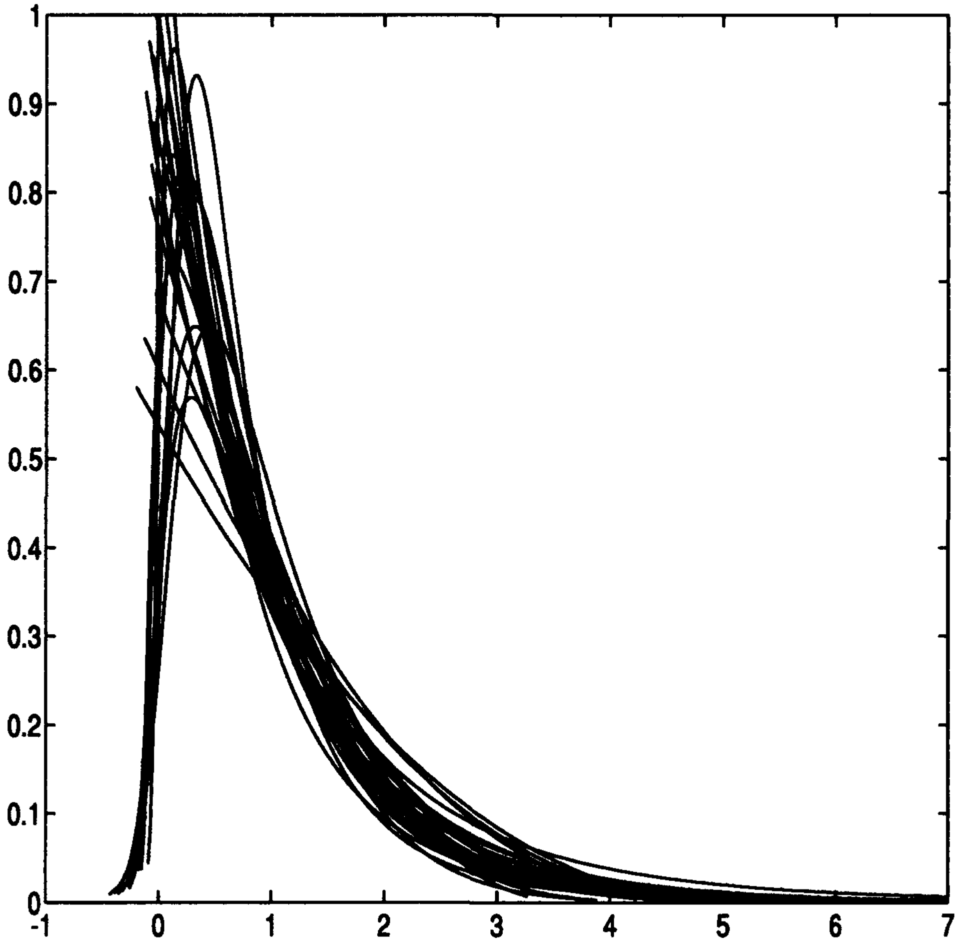

Figure D.2 Plots of 50-element samples from an approximate exponential distribution fitted with linear moments

Monte Carlo Experiment Results

Statistical Summary

Experiment Result		Statistic			
		Minimum	Maximum	Average	Variance
Moments of Sample Data	1st	0.76118800	1.29035000	0.94777557	0.01100065
	2nd	0.36137600	0.66616900	0.48157510	0.00302355
	3rd	0.24242200	0.40198900	0.32050903	0.00173883
	4th	0.09137160	0.24539800	0.15242416	0.00109253
Lambda Parameters	λ_1	-0.11582100	0.15770800	-0.00874052	0.00410422
	λ_2	-0.30981200	0.17745500	0.01700511	0.00996706
	λ_3	-0.00982581	0.00567225	-0.00018382	0.00000784
	λ_4	-0.19996100	0.22428400	0.02720114	0.00850612
Theoretical K-S Statistics	MIN	0.00000037	0.00635650	0.00041018	0.00000134
	AVG	0.00609105	0.04133410	0.01449844	0.00006345
	MAX	0.01750830	0.08462420	0.04547370	0.00038172
Empirical K-S Statistics	MIN	0.00000000	0.00012106	0.00003004	0.00000000
	AVG	0.00473884	0.01318650	0.00855294	0.00000501
	MAX	0.02954890	0.08860690	0.04811859	0.00021221

Experiment Number: 3C

Samples: 30 samples of 100 elements each

Original function: lambda approximation of an exponential function

Method of fitment: method of linear moments

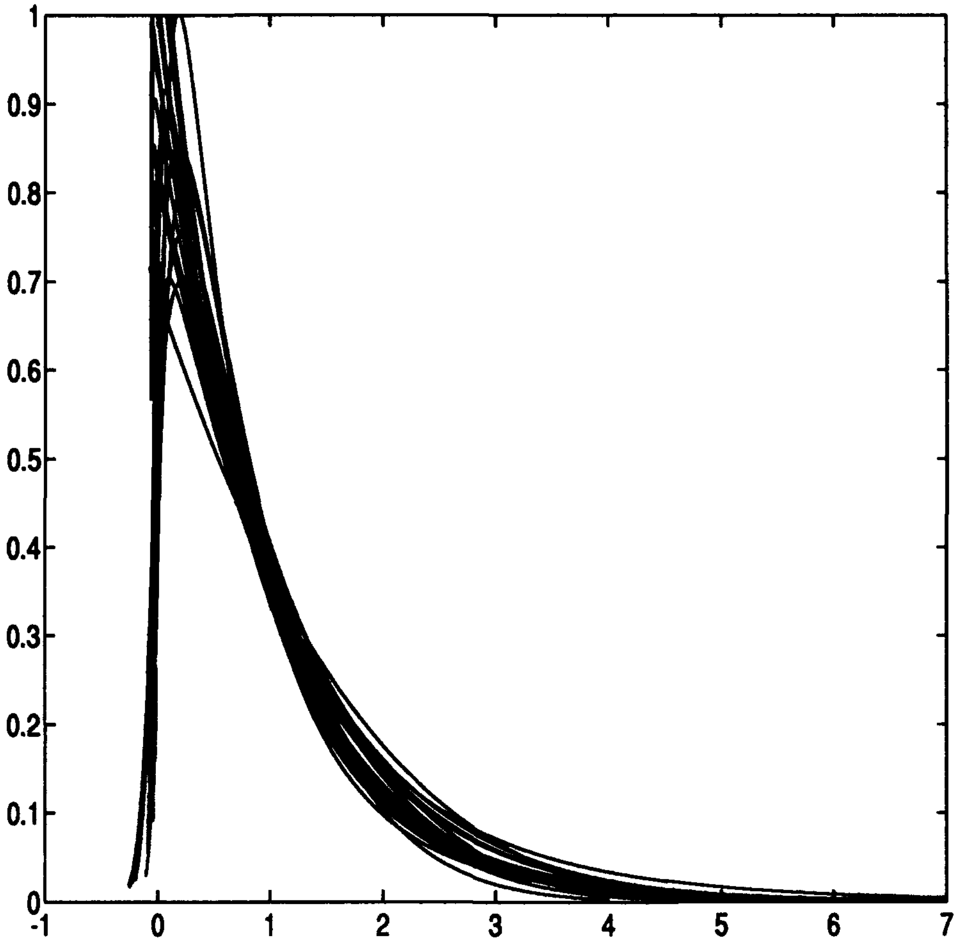

Figure D.3 Plots of 100-element samples from an approximate exponential distribution fitted with linear moments

Appendix E. Experiment Results: Normal Distribution, Conventional Moments

Monte Carlo Experiment Results

Statistical Summary

Experiment Result		Minimum	Maximum	Average	Variance
Moments of Sample Data	1st	-0.32920000	0.33291600	-0.08192267	0.03075791
	2nd	0.68745300	1.54415000	1.03898053	0.05250603
	3rd	-0.98539500	0.63038300	-0.05117874	0.16026090
	4th	1.83407000	4.16280000	2.67890867	0.30904123
Lambda Parameters	λ_1	-1.83070000	1.52185000	-0.14659537	0.88960911
	λ_2	0.07862730	0.38893800	0.24267934	0.00508416
	λ_3	0.00000526	0.54493000	0.20546724	0.02714083
	λ_4	0.00562861	1.16183000	0.25439328	0.07261522
Theoretical K-S Statistics	MIN	0.00000242	0.00132966	0.00022295	0.00000013
	AVG	0.01538140	0.05529040	0.03547728	0.00015160
	MAX	0.04246670	0.16926800	0.09836652	0.00104047
Empirical K-S Statistics	MIN	0.00000000	0.00008005	0.00001482	0.00000000
	AVG	0.01102580	0.03059050	0.02146268	0.00003285
	MAX	0.05243800	0.14790400	0.10000633	0.00054649

Experiment Number: 4A

Samples: 30 samples of 25 elements each

Original function: lambda approximation of a normal function

Method of fitment: conventional method of moments

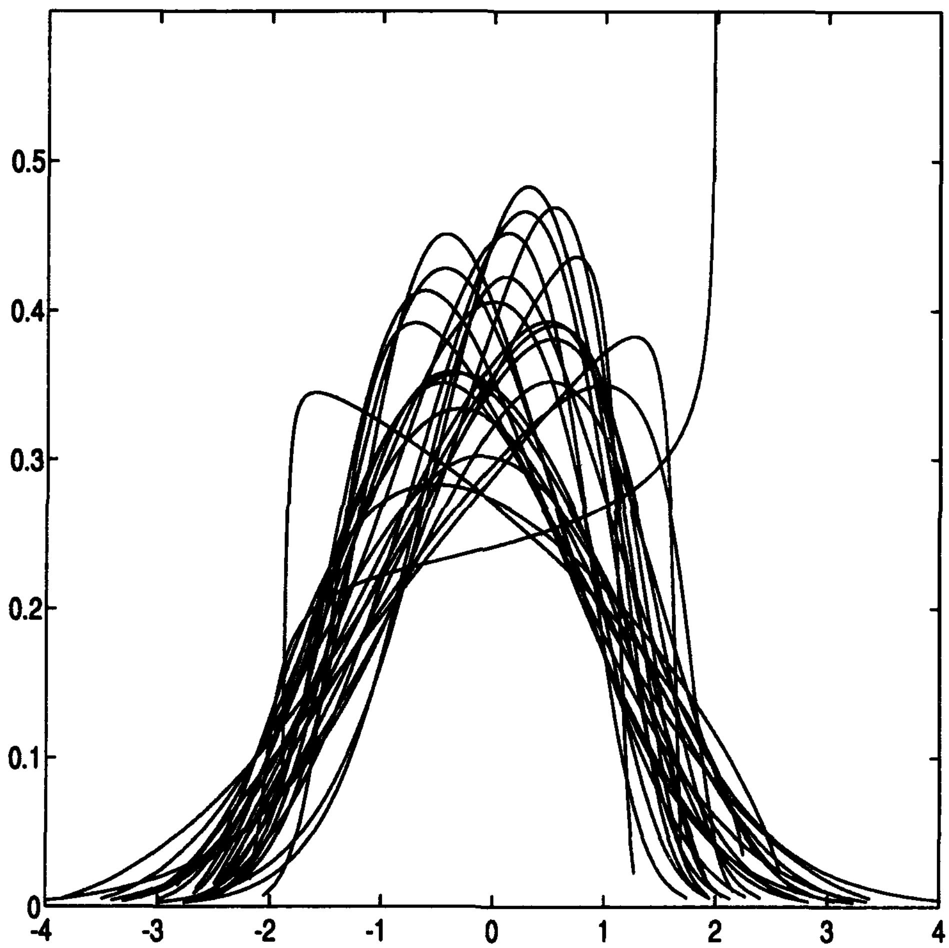

Figure E.1 **Plots of 25-element samples from an approximate normal distribution fitted with conventional moments**

Monte Carlo Experiment Results

Statistical Summary

Experiment Result		Statistic			
		Minimum	Maximum	Average	Variance
Moments of Sample Data	1st	-0.24640400	0.34302800	-0.03126540	0.02190964
	2nd	0.67163200	1.32877000	1.01719943	0.03144692
	3rd	-0.70625700	0.28223700	-0.12639136	0.06207302
	4th	2.07060000	3.46496000	2.70478833	0.12615150
Lambda Parameters	λ_1	-1.20415000	1.24473000	0.06923864	0.34558201
	λ_2	0.10878000	0.38976400	0.25059593	0.00405041
	λ_3	0.06610440	0.45249000	0.21968641	0.01388314
	λ_4	0.03763780	1.03470000	0.21087366	0.04743812
Theoretical K-S Statistics	MIN	0.00000043	0.00132966	0.00017865	0.00000012
	AVG	0.01055100	0.05653010	0.02521661	0.00014575
	MAX	0.02042070	0.12527700	0.06705405	0.00100233
Empirical K-S Statistics	MIN	0.00000000	0.00022214	0.00003048	0.00000000
	AVG	0.01003980	0.03600580	0.01576388	0.00002948
	MAX	0.04035370	0.15387900	0.07544494	0.00061213

Experiment Number: 4B

Samples: 30 samples of 50 elements each

Original function: lambda approximation of a normal function

Method of fitment: conventional method of moments

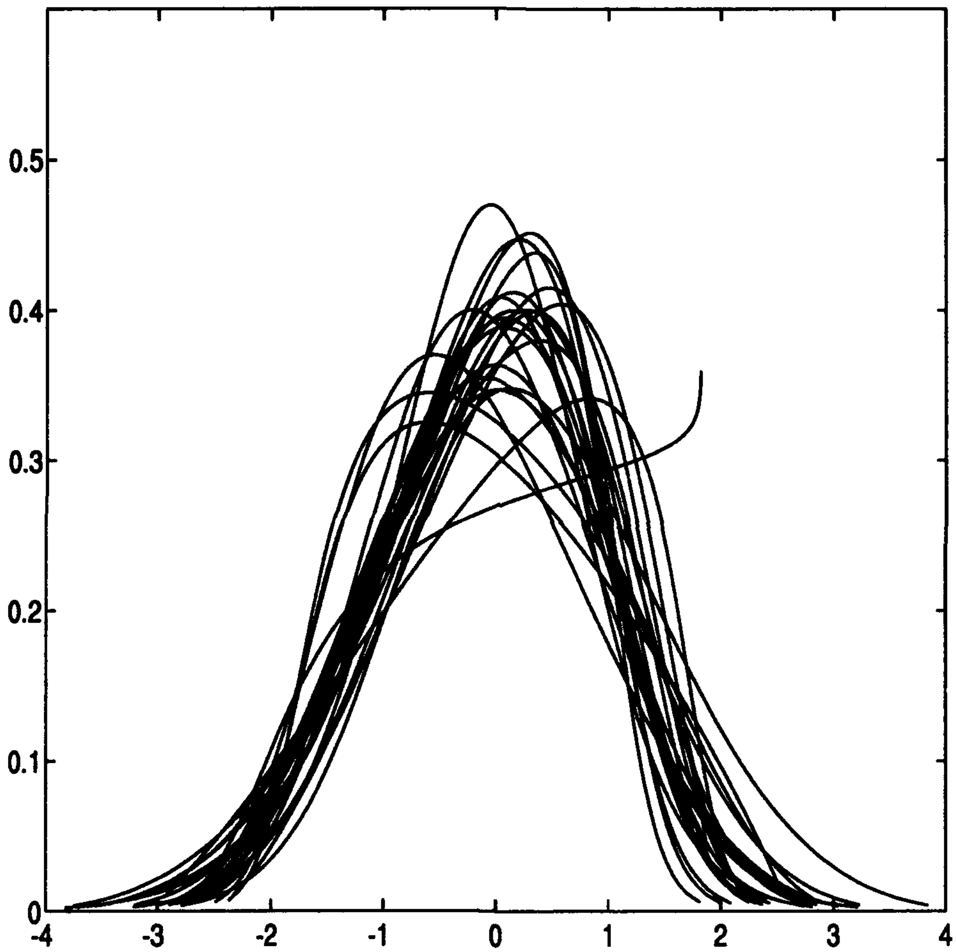

Figure E.2 Plots of 50-element samples from an approximate normal distribution fitted with conventional moments

Monte Carlo Experiment Results

Statistical Summary

Experiment Result		Statistic			
		Minimum	Maximum	Average	Variance
Moments of Sample Data	1st	-0.21766800	0.23810900	-0.04653627	0.01058862
	2nd	0.79556500	1.19861000	0.99209797	0.00903663
	3rd	-0.57158700	0.23436100	-0.04420776	0.03683652
	4th	2.06409000	3.64464000	2.76636533	0.09748634
Lambda Parameters	λ_1	-1.17677000	0.91700300	-0.04620887	0.14842199
	λ_2	0.09009060	0.39928400	0.25147062	0.00426265
	λ_3	0.04436510	0.38893000	0.18841034	0.00562870
	λ_4	0.05461550	0.70557900	0.20469460	0.01811967
Theoretical K-S Statistics	MIN	0.00000005	0.00132745	0.00019018	0.00000015
	AVG	0.00780327	0.03940710	0.01836428	0.00007217
	MAX	0.01603560	0.08966780	0.04879070	0.00048041
Empirical K-S Statistics	MIN	0.00000000	0.00008975	0.00002126	0.00000000
	AVG	0.00642899	0.02660000	0.01141977	0.00001761
	MAX	0.02808660	0.08697620	0.05236926	0.00024790

Experiment Number: 4C

Samples: 30 samples of 100 elements each

Original function: lambda approximation of a normal function

Method of fitment: conventional method of moments

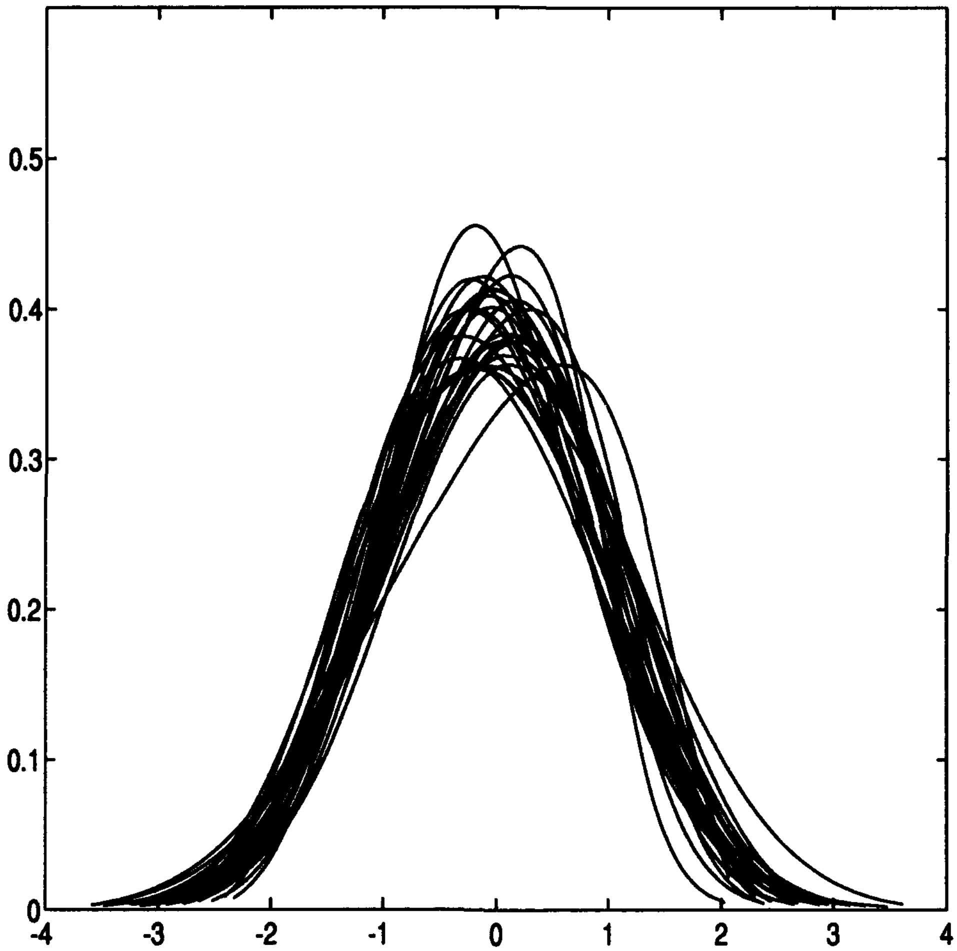

Figure E.3 Plots of 100-element samples from an approximate normal distribution fitted with conventional moments

Appendix F. Experiment Results: Gamma Distribution, Conventional Moments

Monte Carlo Experiment Results

Statistical Summary

Experiment Result		Statistic			
		Minimum	Maximum	Average	Variance
Moments of Sample Data	1st	0.48061600	1.17188000	0.74208173	0.03016900
	2nd	0.44652400	1.83663000	0.93963863	0.11671945
	3rd	0.27005400	2.16257000	0.99939517	0.27040342
	4th	1.95383000	8.77330000	3.77792533	2.94413918
Lambda Parameters	λ_1	-0.77041100	0.15703100	-0.35654108	0.06580892
	λ_2	0.00431302	0.38850300	0.19268309	0.00889336
	λ_3	0.00000020	0.09125090	0.01315165	0.00059144
	λ_4	0.00393935	0.66545900	0.29161819	0.02784793
Theoretical K-S Statistics	MIN	0.00000073	0.00088285	0.00023834	0.00000007
	AVG	0.01365960	0.05092160	0.02837313	0.00011045
	MAX	0.03297290	0.16598500	0.09354674	0.00103906
Empirical K-S Statistics	MIN	0.00000000	0.00039474	0.00002796	0.00000001
	AVG	0.00998735	0.04265360	0.01898630	0.00005508
	MAX	0.05278450	0.18413800	0.11344866	0.00095563

Experiment Number: 5A

Samples: 30 samples of 25 elements each

Original function: lambda approximation of a gamma function

Method of fitment: conventional method of moments

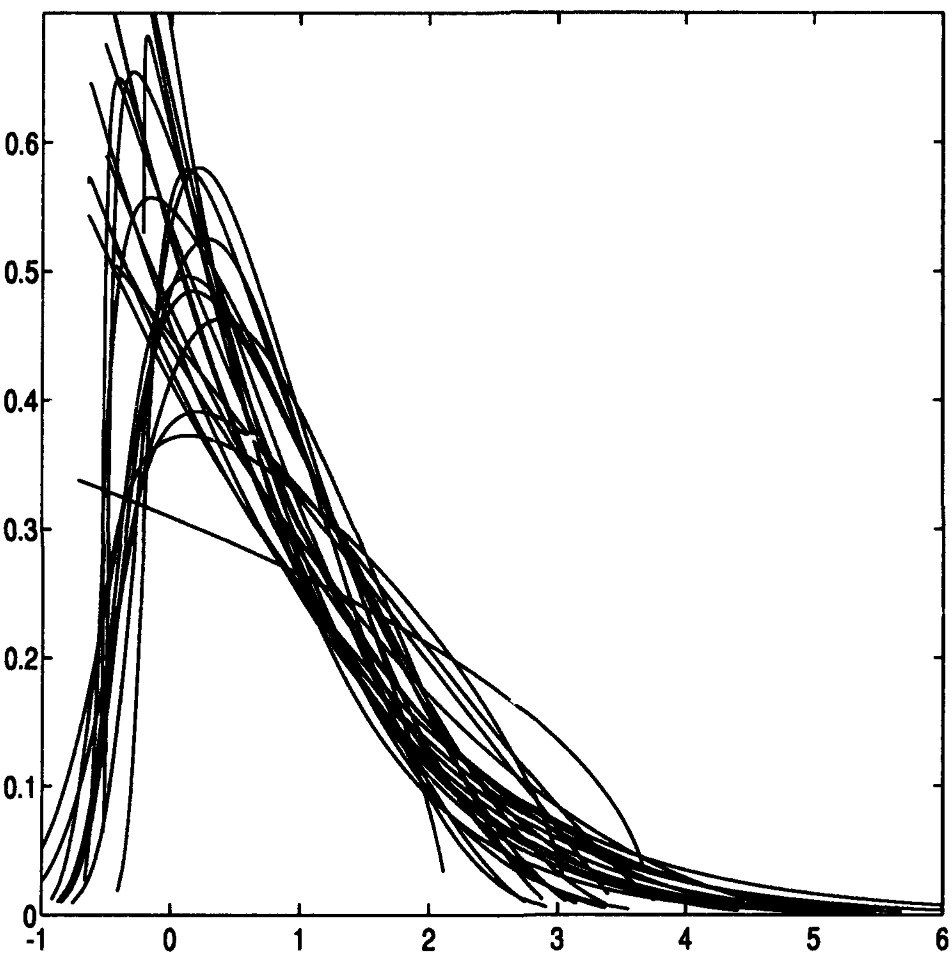

Figure F.1 Plots of 25-element samples from an approximate gamma distribution fitted with conventional moments

Monte Carlo Experiment Results

Statistical Summary

Experiment Result		Statistic			
		Minimum	Maximum	Average	Variance
Moments of Sample Data	1st	0.55609200	1.23099000	0.76728553	0.01912877
	2nd	0.51725800	1.95520000	0.90982420	0.08046629
	3rd	0.43535500	1.50274000	0.97441073	0.10478642
	4th	2.22859000	5.88553000	3.78924267	1.14210294
Lambda Parameters	λ_1	-0.63880000	0.18232200	-0.25121066	0.05703032
	λ_2	0.04615670	0.34312800	0.17102693	0.00565214
	λ_3	0.00000032	0.06328690	0.01115015	0.00019902
	λ_4	0.04424930	0.55072200	0.24769920	0.02387134
Theoretical K-S Statistics	MIN	0.00000002	0.00108552	0.00026016	0.00000010
	AVG	0.00847473	0.06912110	0.02318812	0.00011734
	MAX	0.02679400	0.17418700	0.07323483	0.00092128
Empirical K-S Statistics	MIN	0.00000000	0.00025278	0.00002620	0.00000000
	AVG	0.00712015	0.02764040	0.01552219	0.00003268
	MAX	0.05078420	0.14837300	0.08982747	0.00079356

Experiment Number: 5B

Samples: 30 samples of 50 elements each

Original function: lambda approximation of a gamma function

Method of fitment: conventional method of moments

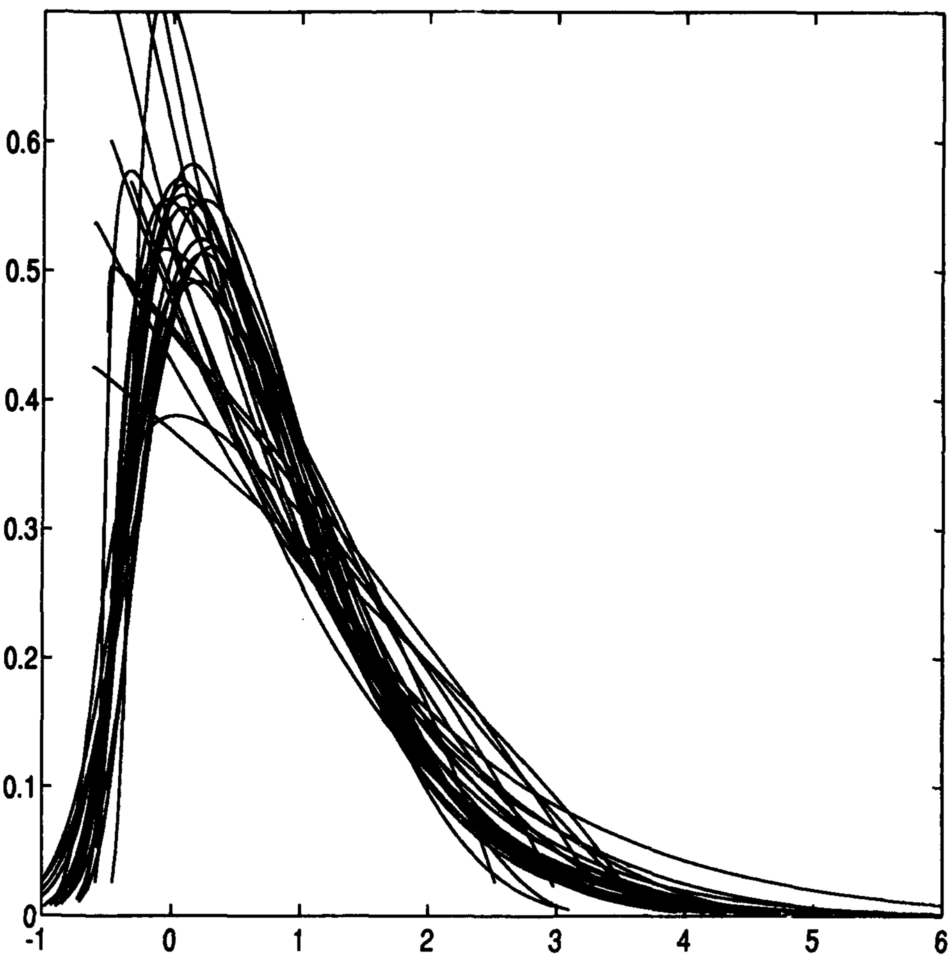

Figure F.2 Plots of 50-element samples from an approximate gamma distribution fitted with conventional moments

Monte Carlo Experiment Results

Statistical Summary

Experiment Result		Statistic			
		Minimum	Maximum	Average	Variance
Moments of Sample Data	1st	0.57285600	1.09224000	0.76092617	0.01117079
	2nd	0.55903500	1.59726000	0.92062887	0.03486003
	3rd	0.62038900	1.84820000	1.14948247	0.08927224
	4th	2.76508000	7.84228000	4.45436900	1.66842836
Lambda Parameters	λ_1	-0.54812700	0.11526300	-0.22124468	0.03425355
	λ_2	0.00056224	0.25148400	0.13620528	0.00450865
	λ_3	0.00000012	0.04750490	0.00918425	0.00014064
	λ_4	0.00049871	0.39069200	0.17601228	0.00988754
Theoretical K-S Statistics	MIN	0.00000137	0.00071150	0.00012467	0.00000004
	AVG	0.00693039	0.04841000	0.01740729	0.00007832
	MAX	0.01557750	0.13230700	0.05959370	0.00089597
Empirical K-S Statistics	MIN	0.00000000	0.00013362	0.00002324	0.00000000
	AVG	0.00691078	0.02198690	0.01272067	0.00001652
	MAX	0.03848610	0.10369800	0.07009377	0.00039885

Experiment Number: 5C

Samples: 30 samples of 100 elements each

Original function: lambda approximation of a gamma function

Method of fitment: conventional method of moments

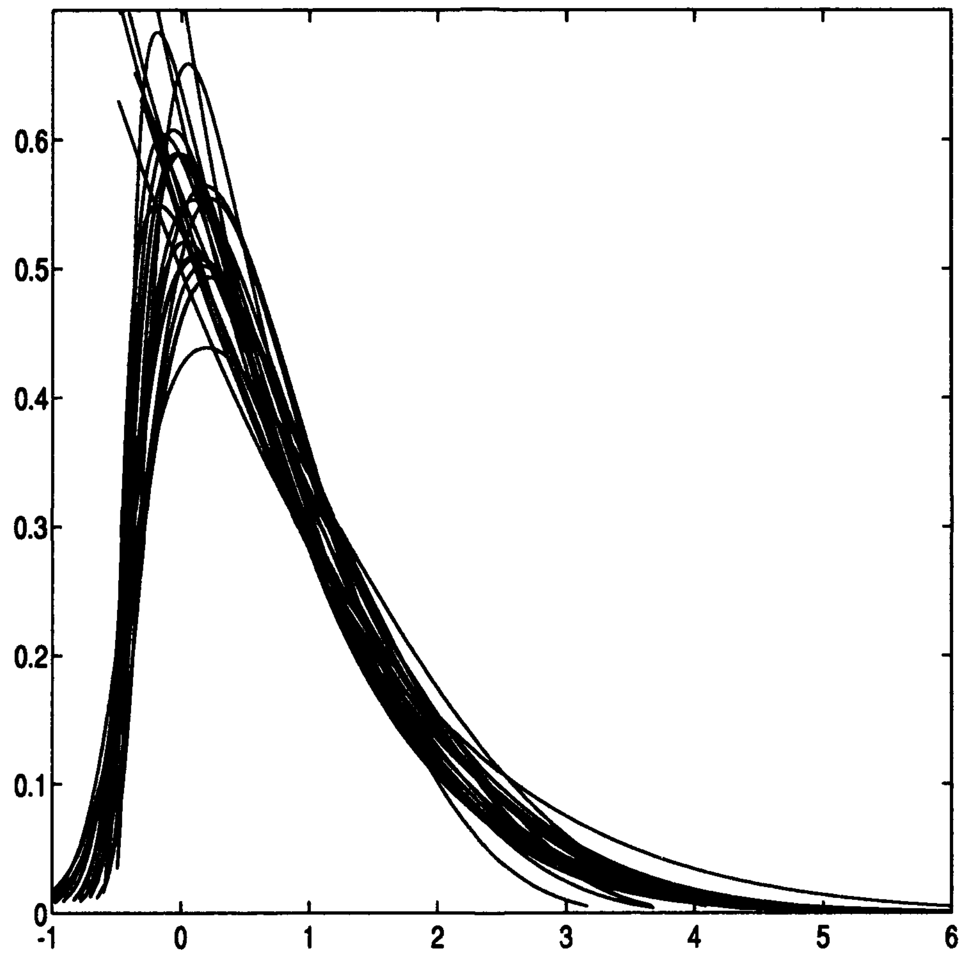

Figure F.3 Plots of 100-element samples from an approximate gamma distribution fitted with conventional moments

Appendix G. Experiment Results: Exponential Distribution, Conventional Moments

Monte Carlo Experiment Results

Statistical Summary

Experiment Result		Statistic			
		Minimum	Maximum	Average	Variance
Moments of Sample Data	1st	0.67087600	1.36818000	0.93589840	0.02967292
	2nd	0.33428800	2.25004000	0.90130390	0.17730260
	3rd	0.59918000	2.66973000	1.37864207	0.30437496
	4th	2.14681000	11.12120000	4.60744700	5.42776605
Lambda Parameters	λ_1	-0.37838800	1.85005000	0.11786311	0.24590210
	λ_2	-0.04330980	0.94384000	0.15224372	0.03758558
	λ_3	-0.00059758	2.73062000	0.17564726	0.44691656
	λ_4	-0.03724520	0.74914900	0.19003573	0.04033796
Theoretical K-S Statistics	MIN	0.00000012	0.00091408	0.00035097	0.00000012
	AVG	0.00890602	0.05396200	0.02693338	0.00013563
	MAX	0.03926670	0.19467400	0.11733060	0.00155823
Empirical K-S Statistics	MIN	0.00000000	0.00057285	0.00004606	0.00000001
	AVG	0.00919480	0.03766350	0.01982872	0.00005756
	MAX	0.05754620	0.19870600	0.13499841	0.00137565

Experiment Number: 6A

Samples: 30 samples of 25 elements each

Original function: lambda approximation of an exponential function

Method of fitment: conventional method of moments

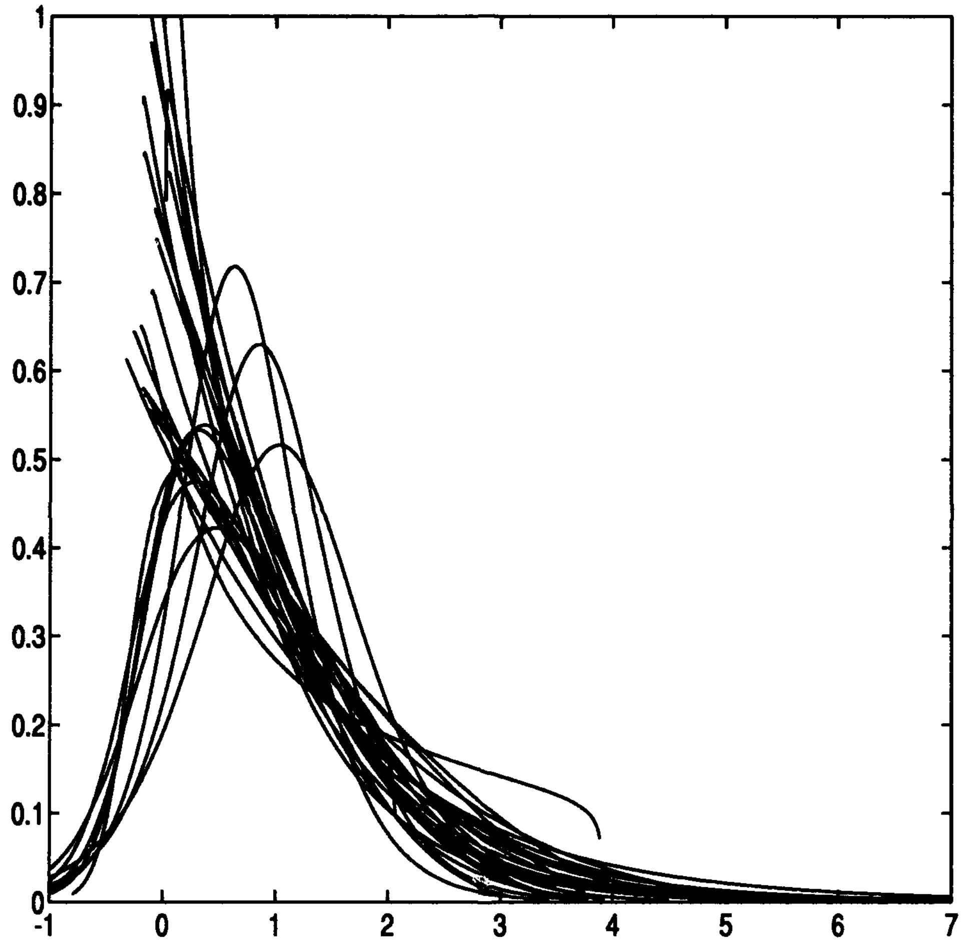

Figure G.1 Plots of 25-element samples from an approximate exponential distribution fitted with conventional moments

Monte Carlo Experiment Results

Statistical Summary

Experiment Result		Statistic			
		Minimum	Maximum	Average	Variance
Moments of Sample Data	1st	0.75096200	1.44070000	0.95519720	0.01969888
	2nd	0.42602100	2.27778000	0.86712400	0.12255268
	3rd	0.75023400	2.02712000	1.38619813	0.14392488
	4th	2.41646000	8.46280000	4.84441167	2.83071526
Lambda Parameters	λ_1	-0.32591000	0.78509000	0.13776838	0.10003427
	λ_2	-0.00229114	0.27153700	0.09106265	0.01129436
	λ_3	-0.00066755	0.00000306	-0.00014337	0.00000003
	λ_4	-0.00178665	0.45000500	0.13092729	0.02572388
Theoretical K-S Statistics	MIN	0.00000578	0.00186403	0.00032063	0.00000018
	AVG	0.00646737	0.07149220	0.02238879	0.00013138
	MAX	0.04346470	0.17194900	0.09520559	0.00112353
Empirical K-S Statistics	MIN	0.00000000	0.00019934	0.00002672	0.00000000
	AVG	0.00599659	0.02537560	0.01523655	0.00002695
	MAX	0.05326360	0.18218700	0.10607955	0.00109799

Experiment Number: 6B

Samples: 30 samples of 50 elements each

Original function: lambda approximation of an exponential function

Method of fitment: conventional method of moments

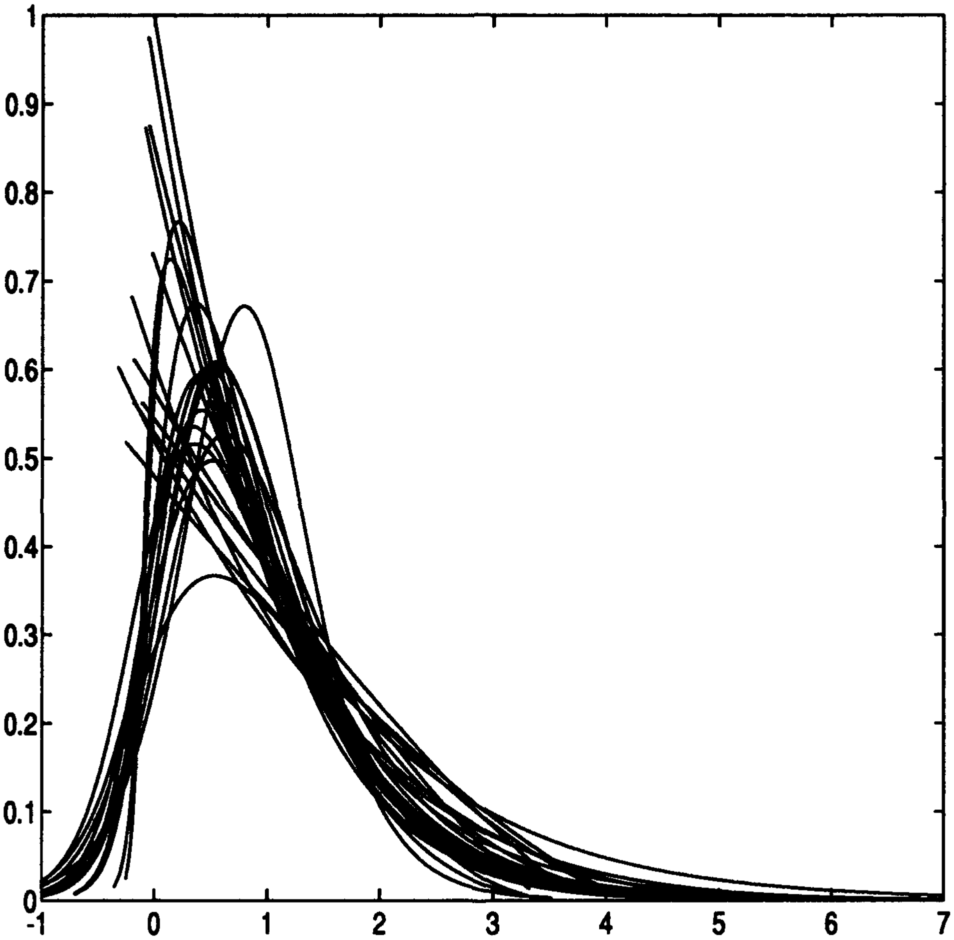

Figure G.2 Plots of 50-element samples from an approximate exponential distribution fitted with conventional moments

Monte Carlo Experiment Results

Statistical Summary

Experiment Result		Statistic			
		Minimum	Maximum	Average	Variance
Moments of Sample Data	1st	0.76118800	1.29035000	0.94777557	0.01100065
	2nd	0.44768600	1.77345000	0.88839270	0.05455836
	3rd	1.05478000	2.40918000	1.59824167	0.13811616
	4th	3.27297000	11.77830000	5.90627500	4.32608383
Lambda Parameters	λ_1	-0.24337700	0.62057600	0.14862391	0.06535422
	λ_2	-0.05348860	0.22742200	0.05582116	0.00799021
	λ_3	-0.00045597	0.00000376	-0.00010683	0.00000002
	λ_4	-0.04584390	0.29532000	0.07217256	0.01275286
Theoretical K-S Statistics	MIN	0.00000022	0.00091408	0.00015843	0.00000006
	AVG	0.00800442	0.04840770	0.01663250	0.00006246
	MAX	0.02207040	0.12617800	0.07962476	0.00095739
Empirical K-S Statistics	MIN	0.00000000	0.00019789	0.00002820	0.00000000
	AVG	0.00508115	0.02284340	0.01257003	0.00001738
	MAX	0.03472890	0.14404300	0.08528730	0.00082612

Experiment Number: 6C

Samples: 30 samples of 100 elements each

Original function: lambda approximation of an exponential function

Method of fitment: conventional method of moments

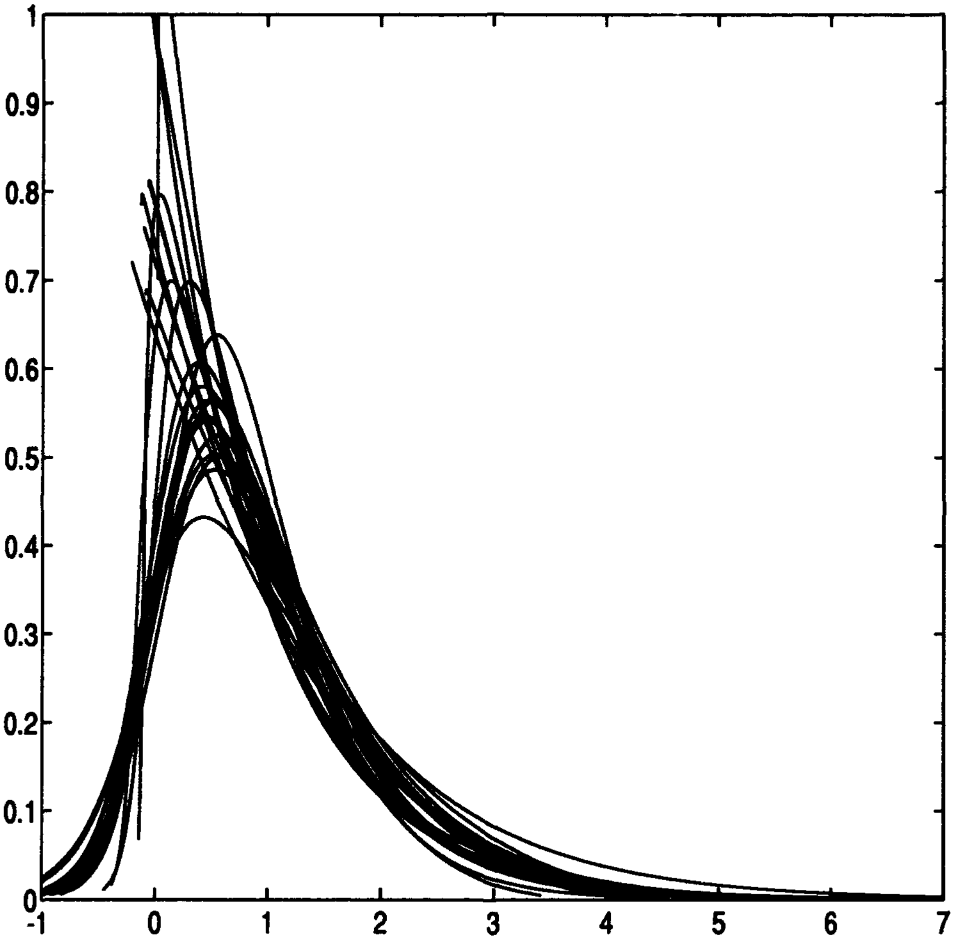

Figure G.3 Plots of 100-element samples from an approximate exponential distribution fitted with conventional moments

Appendix H. Experiment Results: Normal Distribution, Alternate Moments

Monte Carlo Experiment Results

Statistical Summary

Experiment Result		Statistic			
		Minimum	Maximum	Average	Variance
Moments of Sample Data	1st	-0.32920000	0.33291600	-0.08192267	0.03075791
	2nd	0.68745300	1.54415000	1.03898053	0.05250603
	3rd	0.46901900	1.74990000	1.01318137	0.12075743
	4th	1.87260000	3.05616000	2.42375433	0.07598671
Lambda Parameters	λ_1	-1.81571000	2.08373000	0.01490352	0.78299881
	λ_2	-0.15520400	0.40728500	0.21428818	0.01661824
	λ_3	-0.06558470	0.84846300	0.22445681	0.04732022
	λ_4	-0.08777880	0.61003900	0.18532574	0.04073945
Theoretical K-S Statistics	MIN	0.00000194	0.00132908	0.00014137	0.00000008
	AVG	0.01242070	0.05460580	0.03524031	0.00015237
	MAX	0.03273380	0.15955000	0.09805817	0.00119411
Empirical K-S Statistics	MIN	0.00000000	0.00013531	0.00001919	0.00000000
	AVG	0.01167520	0.02929620	0.02041950	0.00002269
	MAX	0.05895210	0.14434900	0.09787561	0.00047475

Experiment Number: 7A

Samples: 30 samples of 25 elements each

Original function: lambda approximation of a normal function

Method of fitment: alternate method (Q_3, Q_4)

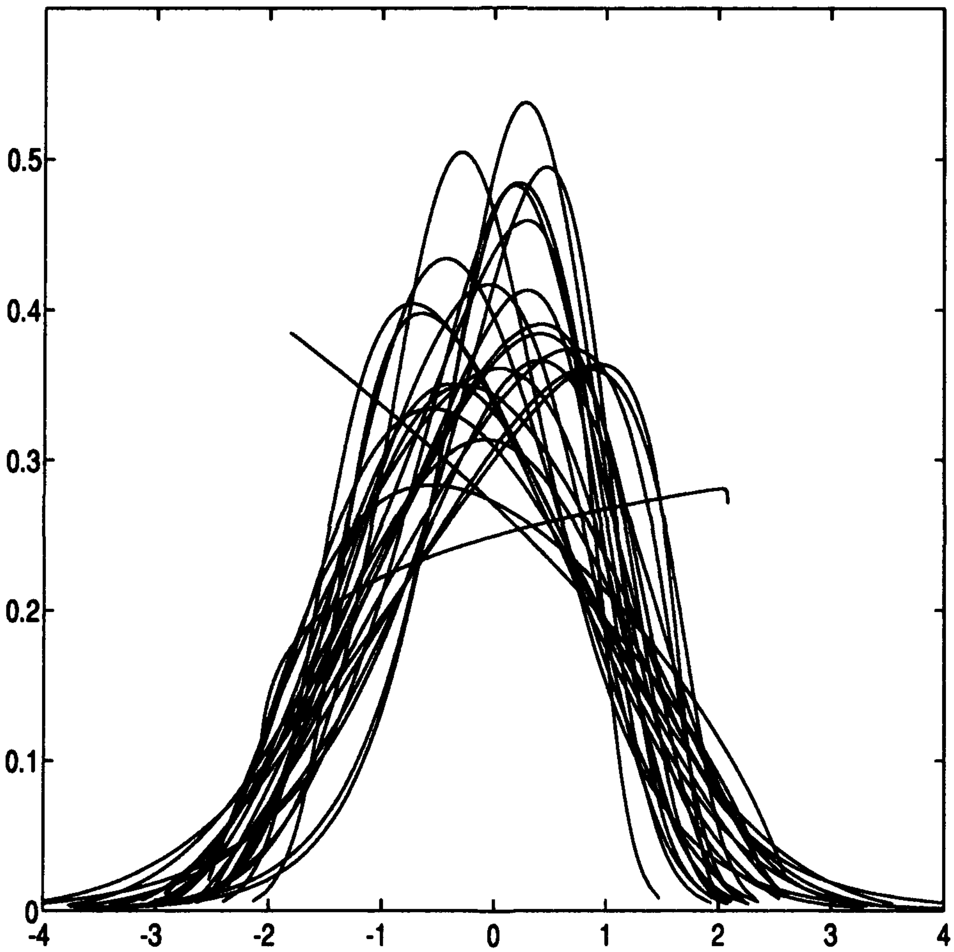

Figure H.1 Plots of 25-element samples from an approximate normal distribution fitted with alternate moments

Monte Carlo Experiment Results

Statistical Summary

Experiment Result		Statistic			
		Minimum	Maximum	Average	Variance
Moments of Sample Data	1st	-0.24640400	0.34302800	-0.04351340	0.01692519
	2nd	0.67163200	1.32877000	1.01043343	0.02903252
	3rd	0.61927500	1.38304000	0.97787483	0.03994153
	4th	1.98703000	2.95123000	2.44442767	0.04833684
Lambda Parameters	λ_1	-1.70150000	1.52504000	-0.03031136	0.54837742
	λ_2	-0.08009890	0.35489200	0.22881436	0.00785004
	λ_3	-0.03850840	0.70451800	0.19743541	0.02759746
	λ_4	-0.03574680	0.76673800	0.20614792	0.04422529
Theoretical K-S Statistics	MIN	0.00000051	0.00128625	0.00018631	0.00000014
	AVG	0.01011840	0.05588620	0.02453645	0.00011004
	MAX	0.02602540	0.12451500	0.06674605	0.00087331
Empirical K-S Statistics	MIN	0.00000000	0.00009787	0.00002051	0.00000000
	AVG	0.00994767	0.02185460	0.01436887	0.00001010
	MAX	0.04597370	0.13449200	0.07069398	0.00036417

Experiment Number: 7B

Samples: 30 samples of 50 elements each

Original function: lambda approximation of a normal function

Method of fitment: alternate method (Q_3, Q_4)

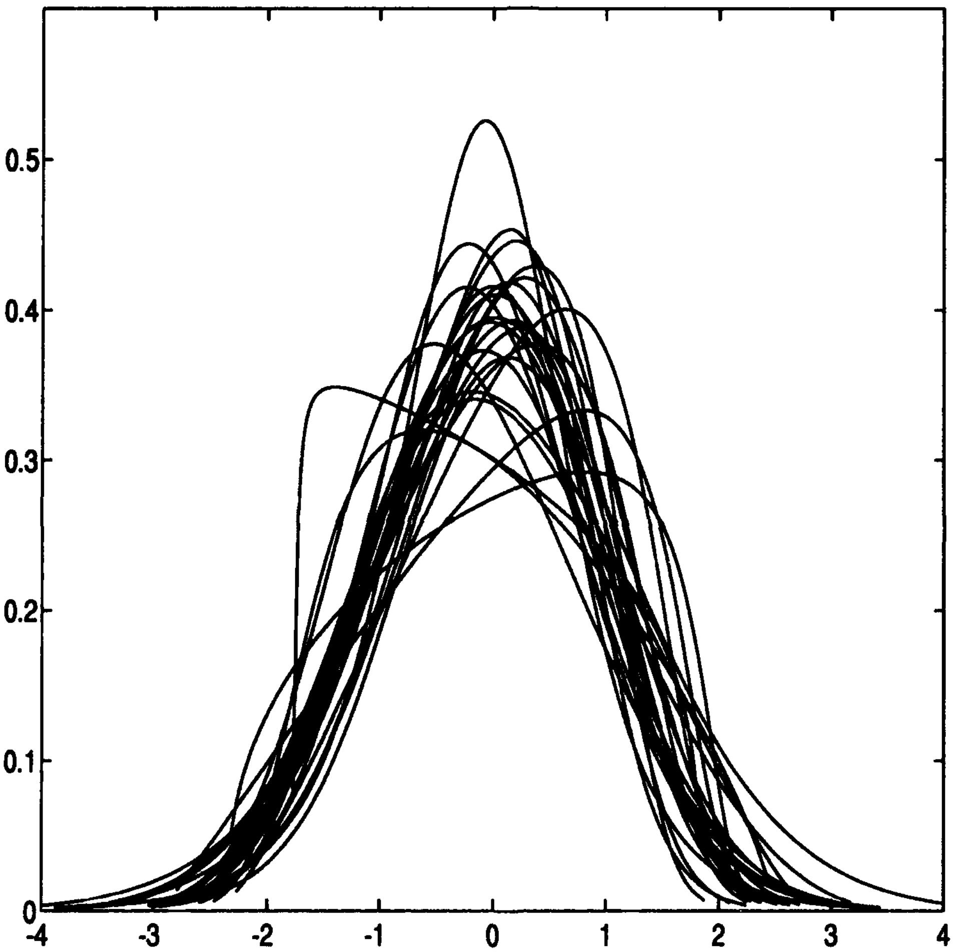

Figure H.2 Plots of 50-element samples from an approximate normal distribution fitted with alternate moments

Monte Carlo Experiment Results

Statistical Summary

Experiment Result		Statistic			
		Minimum	Maximum	Average	Variance
Moments of Sample Data	1st	-0.21766800	0.23810900	-0.04653627	0.01058862
	2nd	0.79556500	1.19861000	0.99209797	0.00903663
	3rd	0.65539800	1.23400000	0.98769230	0.02105896
	4th	2.13103000	2.89916000	2.50212733	0.02084603
Lambda Parameters	λ_1	-1.15944000	0.85998900	-0.04724554	0.17590329
	λ_2	-0.02939990	0.34285700	0.23702967	0.00653531
	λ_3	-0.01272650	0.38912000	0.18013660	0.00888541
	λ_4	-0.01722070	0.69886500	0.19117286	0.01827860
Theoretical K-S Statistics	MIN	0.00000028	0.00127119	0.00015771	0.00000012
	AVG	0.00778402	0.03931520	0.01862765	0.00007079
	MAX	0.01747080	0.09863610	0.05115571	0.00055662
Empirical K-S Statistics	MIN	0.00000000	0.00005997	0.00001558	0.00000000
	AVG	0.00640562	0.02018290	0.01038700	0.00000838
	MAX	0.03170860	0.08823640	0.05005300	0.00024256

Experiment Number: 7C

Samples: 30 samples of 100 elements each

Original function: lambda approximation of a normal function

Method of fitment: alternate method (Q_3, Q_4)

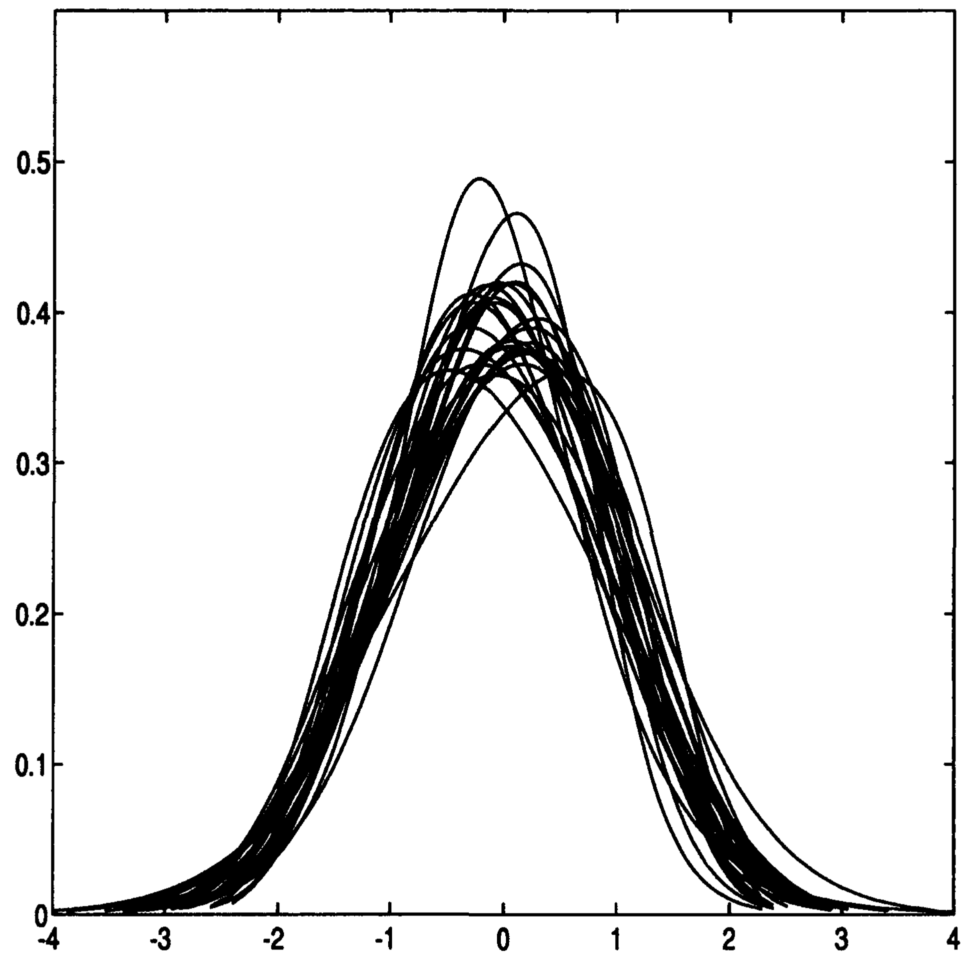

Figure H.3 Plots of 100-element samples from an approximate normal distribution fitted with alternate moments

Appendix I. Experiment Results: Gamma Distribution, Alternate Moments

Monte Carlo Experiment Results

Statistical Summary

Experiment Result		Statistic			
		Minimum	Maximum	Average	Variance
Moments of Sample Data	1st	0.48061600	1.17188000	0.74208173	0.03016900
	2nd	0.44652400	1.83663000	0.93963863	0.11671945
	3rd	1.13009000	4.67810000	2.52182233	0.88282300
	4th	1.85934000	3.51239000	2.55625267	0.13492001
Lambda Parameters	λ_1	-0.71157300	0.41072000	-0.20031334	0.07819155
	λ_2	-0.32067000	0.42483700	0.13612248	0.02632154
	λ_3	-0.03314990	0.14390200	0.01575699	0.00103320
	λ_4	-0.17998800	0.57410200	0.20646465	0.04281049
Theoretical K-S Statistics	MIN	0.00000011	0.00088285	0.00015506	0.00000006
	AVG	0.01037650	0.05087940	0.02823904	0.00010827
	MAX	0.02847320	0.16217900	0.09367535	0.00099154
Empirical K-S Statistics	MIN	0.00000000	0.00049341	0.00003267	0.00000001
	AVG	0.01000580	0.03048770	0.01731186	0.00002533
	MAX	0.06234880	0.16882500	0.10338170	0.00073998

Experiment Number: 8A

Samples: 30 samples of 25 elements each

Original function: lambda approximation of a gamma function

Method of fitment: alternate method (Q_3, Q_4)

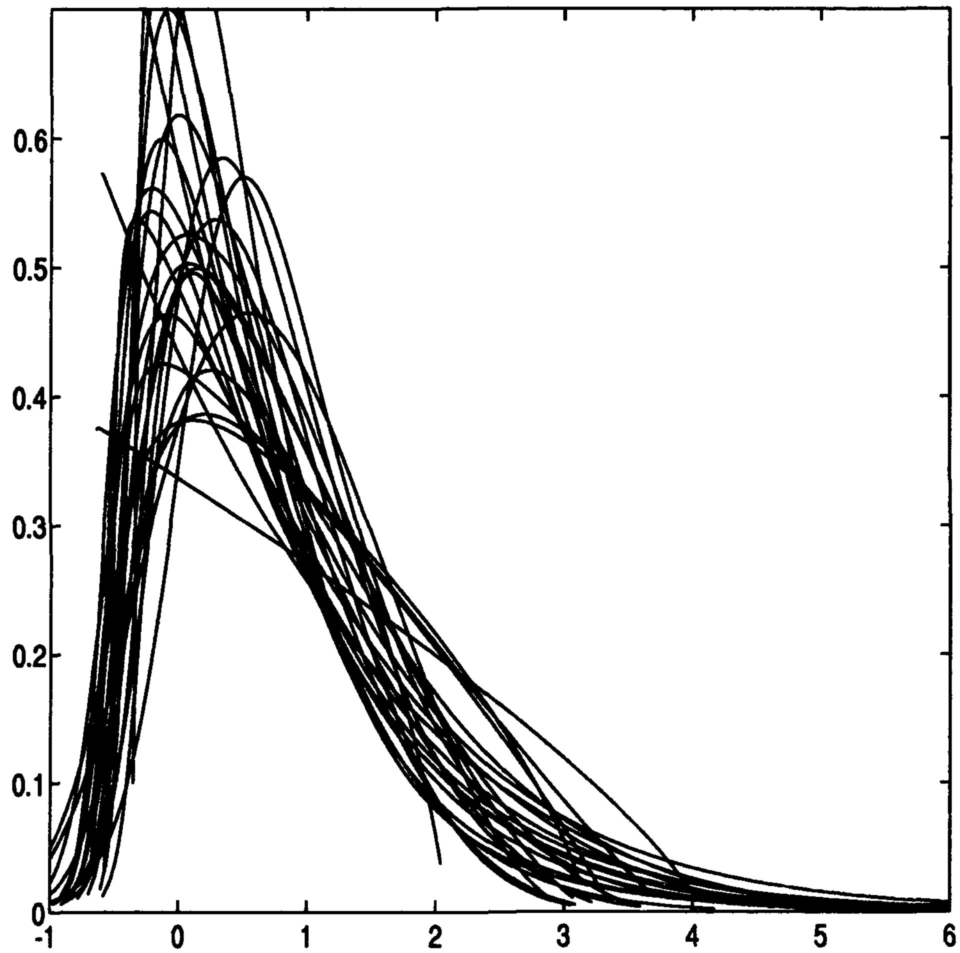

Figure I.1 Plots of 25-element samples from an approximate gamma distribution fitted with alternate moments

Monte Carlo Experiment Results

Statistical Summary

Experiment Result		Statistic			
		Minimum	Maximum	Average	Variance
Moments of Sample Data	1st	0.55609200	1.23099000	0.76728553	0.01912877
	2nd	0.51725800	1.95520000	0.90982420	0.08046629
	3rd	1.53465000	3.49965000	2.45939067	0.31804869
	4th	2.09833000	3.08585000	2.54661433	0.07307762
Lambda Parameters	λ_1	-0.59665300	0.20914800	-0.16800964	0.05185907
	λ_2	-0.18378400	0.34113400	0.13081282	0.01268965
	λ_3	-0.03185830	0.04747990	0.00908726	0.00021125
	λ_4	-0.10452200	0.51679700	0.18775621	0.02943946
Theoretical K-S Statistics	MIN	0.00000032	0.00088285	0.00018192	0.00000008
	AVG	0.00824128	0.05872040	0.02087910	0.00009499
	MAX	0.02166040	0.12191200	0.06534404	0.00081286
Empirical K-S Statistics	MIN	0.00000000	0.00006081	0.00000854	0.00000000
	AVG	0.00676882	0.01756250	0.01199648	0.00000938
	MAX	0.04512980	0.12361500	0.06964856	0.00033589

Experiment Number: 8B

Samples: 30 samples of 50 elements each

Original function: lambda approximation of a gamma function

Method of fitment: alternate method (Q_3, Q_4)

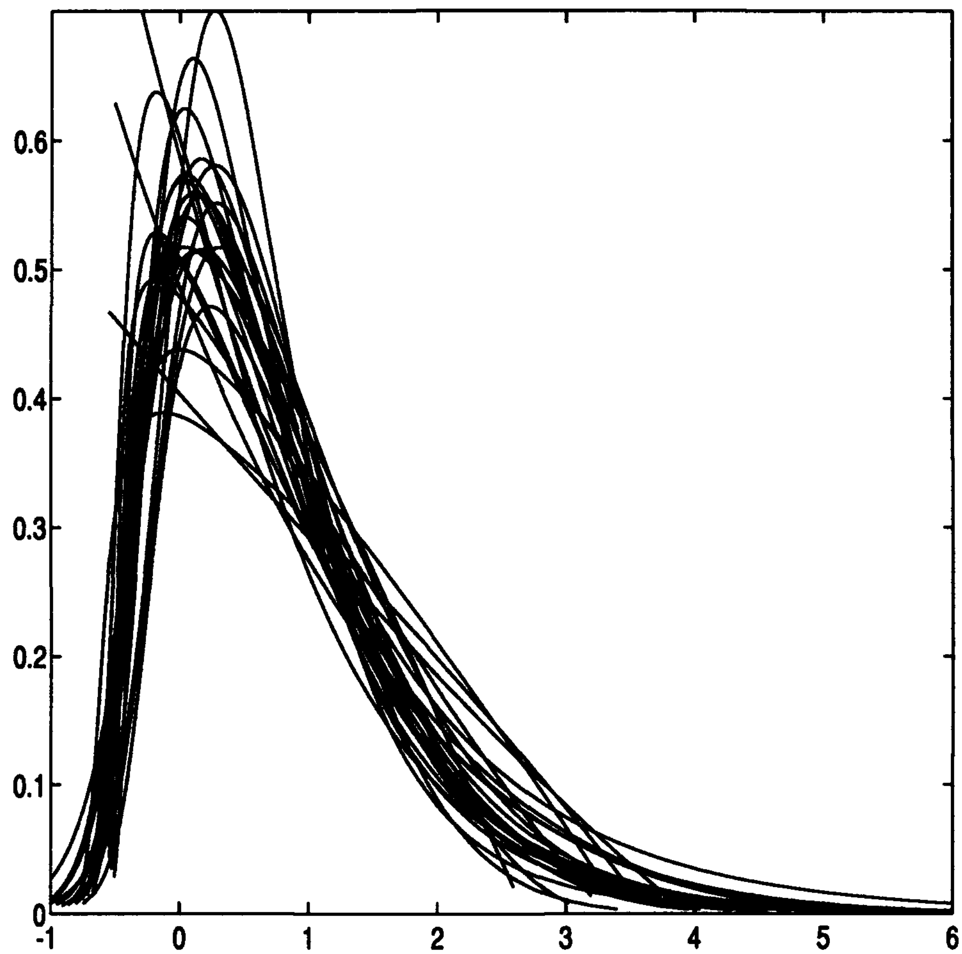

Figure I.2 Plots of 50-element samples from an approximate gamma distribution fitted with alternate moments

Monte Carlo Experiment Results

Statistical Summary

Experiment Result		Statistic			
		Minimum	Maximum	Average	Variance
Moments of Sample Data	1st	0.57285600	1.09224000	0.76092617	0.01117079
	2nd	0.55903500	1.59726000	0.92062887	0.03486003
	3rd	1.62456000	3.39448000	2.53449067	0.18600054
	4th	2.24375000	3.27435000	2.66284800	0.04042513
Lambda Parameters	λ_1	-0.50730100	0.16275400	-0.09489049	0.02077248
	λ_2	-0.29228700	0.22253200	0.09354151	0.01083183
	λ_3	-0.03866660	0.03801980	0.00958450	0.00019587
	λ_4	-0.17385200	0.33751500	0.11798481	0.01265736
Theoretical K-S Statistics	MIN	0.00000053	0.00088285	0.00011864	0.00000006
	AVG	0.00682230	0.04039520	0.01554683	0.00006200
	MAX	0.01639940	0.09027510	0.04910512	0.00045255
Empirical K-S Statistics	MIN	0.00000000	0.00005496	0.00001453	0.00000000
	AVG	0.00507575	0.01403150	0.00871258	0.00000485
	MAX	0.03017440	0.08858540	0.04963251	0.00022438

Experiment Number: 8C

Samples: 30 samples of 100 elements each

Original function: lambda approximation of a gamma function

Method of fitment: alternate method (Q_3, Q_4)

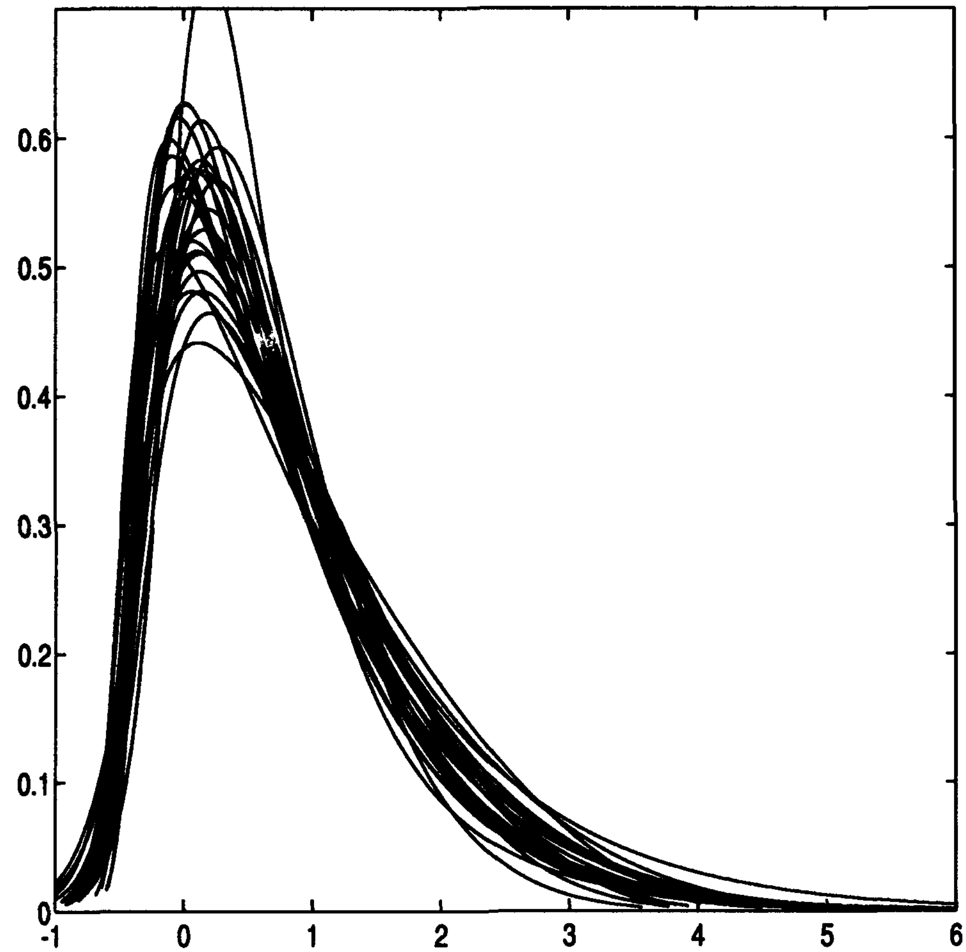

Figure I.3 Plots of 100-element samples from an approximate gamma distribution fitted with alternate moments

Appendix J. Experiment Results: Exponential Distribution, Alternate Moments

Monte Carlo Experiment Results

Statistical Summary

Experiment Result		Statistic			
		Minimum	Maximum	Average	Variance
Moments of Sample Data	1st	0.67087600	1.36818000	0.93589840	0.02967292
	2nd	0.33428800	2.25004000	0.90130390	0.17730260
	3rd	2.21996000	7.41092000	4.21758600	2.46529077
	4th	1.85761000	3.74385000	2.60321133	0.20674032
Lambda Parameters	λ_1	-0.22480400	0.57029400	-0.01852245	0.02455917
	λ_2	-0.49614100	0.35312500	0.05022049	0.03538417
	λ_3	-0.04763650	0.00964982	-0.00115794	0.00008390
	λ_4	-0.30838200	0.40841600	0.09051723	0.03646706
Theoretical K-S Statistics	MIN	0.00000046	0.00872040	0.00057302	0.00000248
	AVG	0.00395157	0.04919680	0.02529038	0.00012810
	MAX	0.03431670	0.23971400	0.08873605	0.00173606
Empirical K-S Statistics	MIN	0.00000000	0.00029412	0.00004913	0.00000001
	AVG	0.00947864	0.02971550	0.01766614	0.00003414
	MAX	0.06381820	0.23657200	0.11254081	0.00130056

Experiment Number: 9A

Samples: 30 samples of 25 elements each

Original function: lambda approximation of an exponential function

Method of fitment: alternate method (Q_3, Q_4)

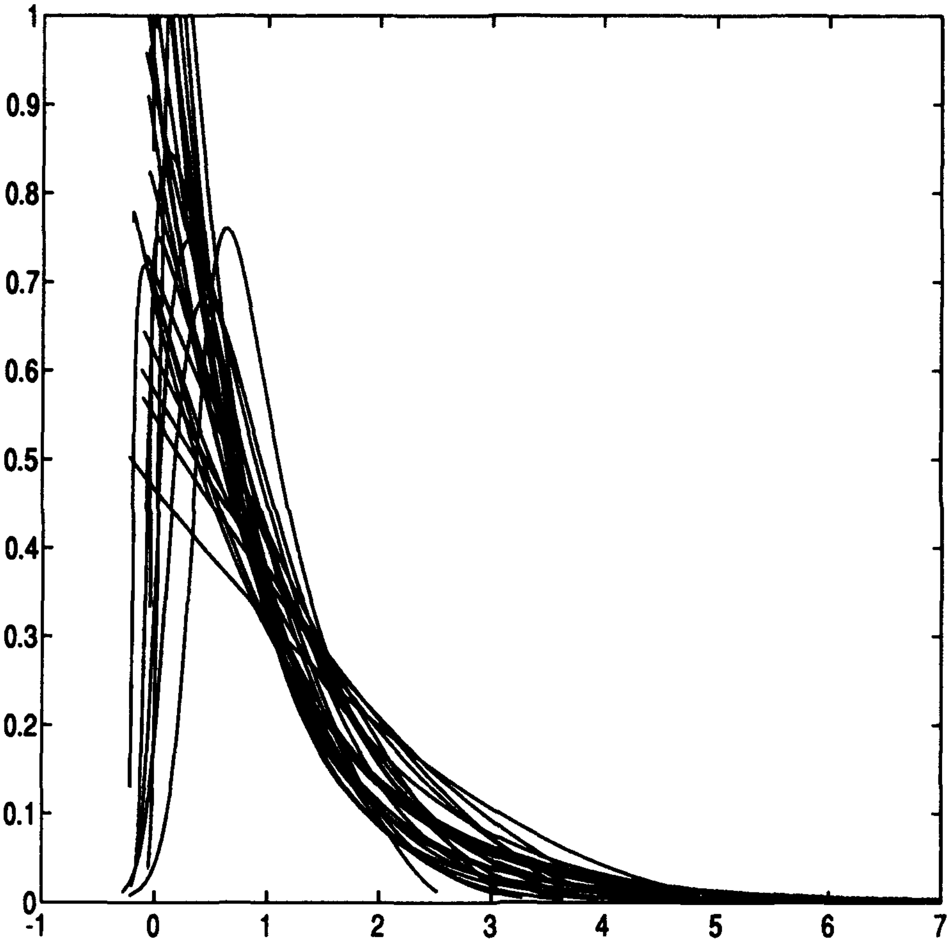

Figure J.1 Plots of 25-element samples from an approximate exponential distribution fitted with alternate moments

Monte Carlo Experiment Results

Statistical Summary

Experiment Result		Statistic			
		Minimum	Maximum	Average	Variance
Moments of Sample Data	1st	0.75096200	1.44070000	0.95519720	0.01969888
	2nd	0.42602100	2.27778000	0.86712400	0.12255268
	3rd	2.54713000	6.81954000	4.17582967	1.11603068
	4th	2.09800000	3.12824000	2.58020400	0.09538596
Lambda Parameters	λ_1	-0.15976300	0.20232000	-0.00059699	0.00590568
	λ_2	-0.18197200	0.27843100	0.05597127	0.01261634
	λ_3	-0.00878814	0.00441249	0.00000719	0.00000366
	λ_4	-0.11115800	0.33144400	0.07108713	0.01512170
Theoretical K-S Statistics	MIN	0.00000022	0.00183874	0.00040423	0.00000019
	AVG	0.00570432	0.06348880	0.01867957	0.00012561
	MAX	0.02510570	0.13498200	0.05966165	0.00081784
Empirical K-S Statistics	MIN	0.00000000	0.00014532	0.00003065	0.00000000
	AVG	0.00585411	0.02032030	0.01239072	0.00001365
	MAX	0.04612920	0.11354700	0.07479089	0.00035223

Experiment Number: 9B

Samples: 30 samples of 50 elements each

Original function: lambda approximation of an exponential function

Method of fitment: alternate method (Q_3, Q_4)

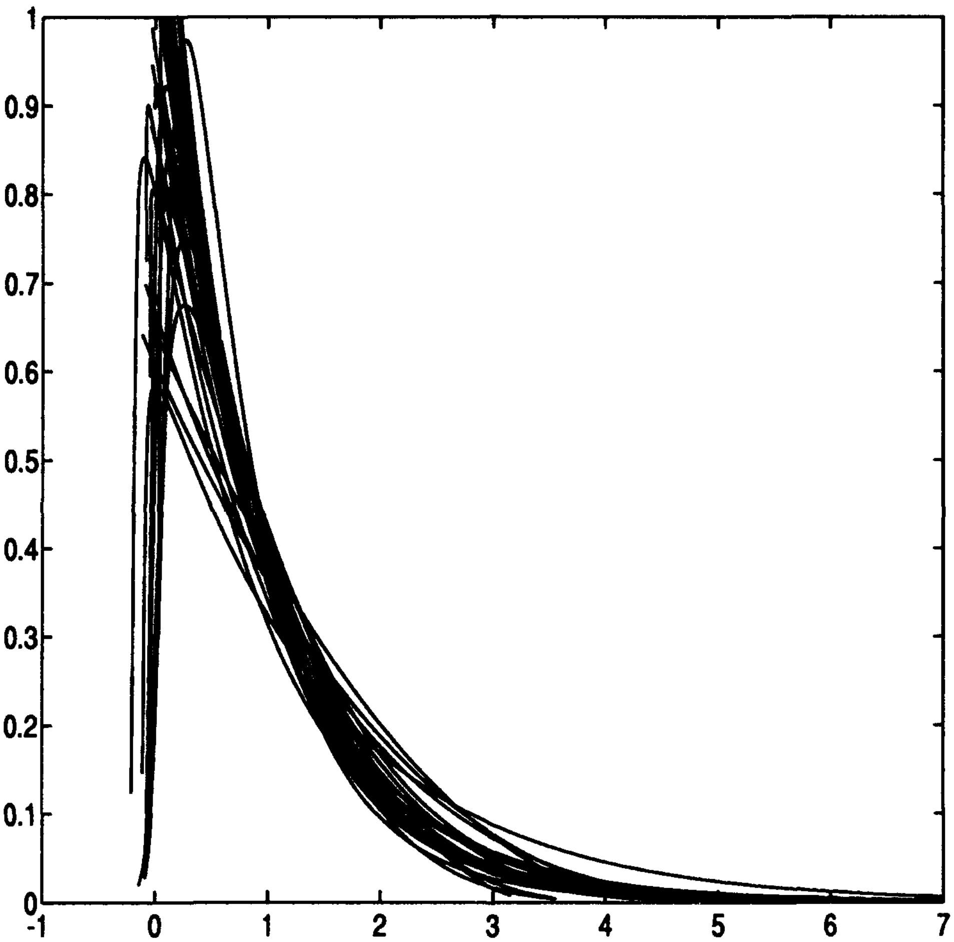

Figure J.2 Plots of 50-element samples from an approximate exponential distribution fitted with alternate moments

Monte Carlo Experiment Results

Statistical Summary

Experiment Result		Minimum	Maximum	Average	Variance
Moments of Sample Data	1st	0.76118800	1.29035000	0.94777557	0.01100065
	2nd	0.44768600	1.77345000	0.88839270	0.05455836
	3rd	3.01858000	6.07224000	4.27603400	0.65647523
	4th	2.29008000	3.43683000	2.72610767	0.05867238
Lambda Parameters	λ_1	-0.09161470	0.11692500	-0.01957376	0.00235948
	λ_2	-0.09522080	0.17726900	0.04257708	0.00597940
	λ_3	-0.00279383	0.00386742	0.00007093	0.00000107
	λ_4	-0.08411940	0.22178400	0.05007467	0.00660096
Theoretical K-S Statistics	MIN	0.00000203	0.00091407	0.00026584	0.00000009
	AVG	0.00306969	0.04246320	0.01428211	0.00006877
	MAX	0.01800200	0.09584070	0.04437378	0.00042776
Empirical K-S Statistics	MIN	0.00000000	0.00011963	0.00002374	0.00000000
	AVG	0.00452803	0.01687420	0.00906632	0.00000789
	MAX	0.02731380	0.10242200	0.05183214	0.00040186

Experiment Number: 9C

Samples: 30 samples of 100 elements each

Original function: lambda approximation of an exponential function

Method of fitment: alternate method (Q_3, Q_4)

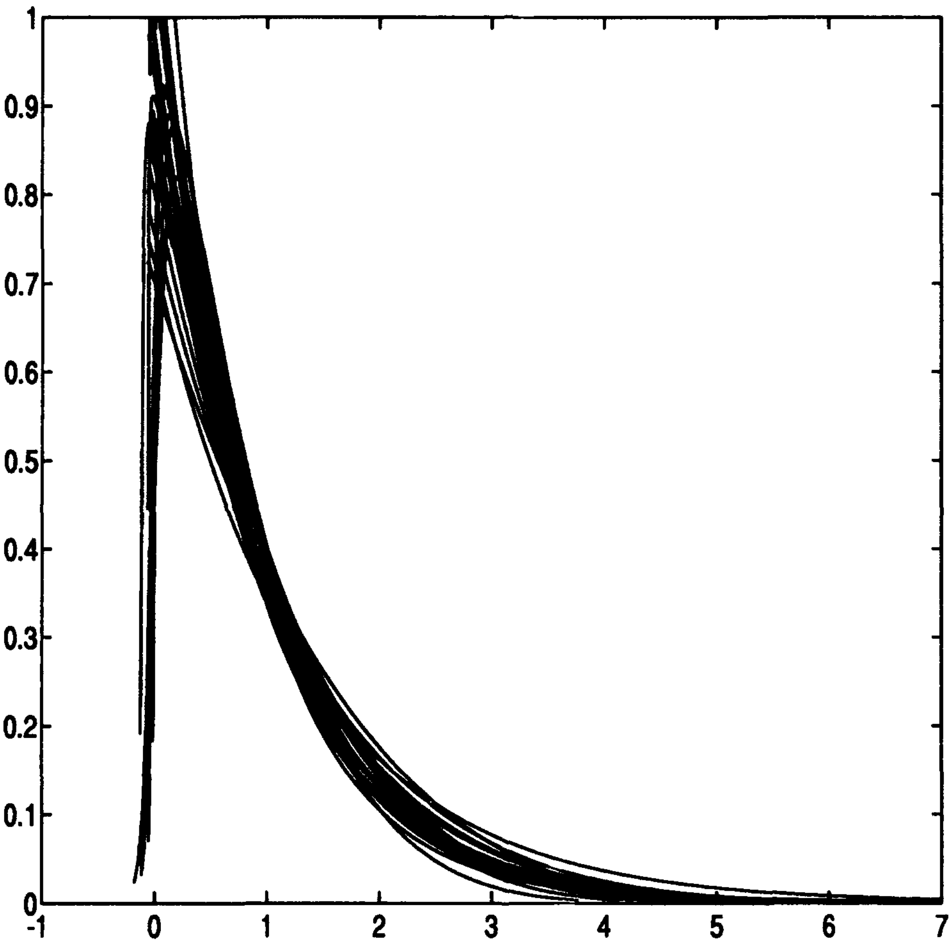

Figure J.3 Plots of 100-element samples from an approximate exponential distribution fitted with alternate moments

Appendix K. FORTRAN Programs: Random Variate Data Sample Generator

```
******************************************************************************
*  FORTRAN 77 CODE USED IN AFIT/ENS MASTER'S THESIS;
*  "THE USE OF L-MOMENTS TO FIT THE GENERALIZED LAMBDA DISTRIBUTION
*   TO SAMPLE DATA"
*  SUBROUTINE TITLE:  SAMPGEN
*  SUBROUTINE PURPOSE:  GENERATE A FILE OF RANDOM VARIATES FROM THE GLD
*                       PERCENTILE FUNCTION
*  SUBROUTINE DESCRIPTION:
*     This program loads an array with a random number generator, generating
*     pseudo-random numbers from zero to one.  Each random variate is then
*     converted to a GLD function using the GLD quantile function, with
*     the lambda values set as parameters of the program.
*     The parameter section of the program provides the programmer with
*     several sets of distribution parameters to choose from.  The programmer
*     chooses the desired distribution by commenting out all the other
*     parameter statements.  For example, in this copy, the NORMAL distribution
*     is selected.
*  R. B. MOHAN, CAPTAIN, USAF
*  DEPARTMENT OF OPERATIONAL SCIENCES
*  GRADUATE SCHOOL OF ENGINEERING, AIR FORCE INSTITUTE OF TECHNOLOGY
*  WRIGHT-PATTERSON AIR FORCE BASE
*  OHIO  45433, USA
*  DATE:  OCTOBER 1993
*  INPUT PARAMETERS:  NONE.
*  OUTPUT PARAMETERS:  NONE.
******************************************************************************
      SUBROUTINE SAMPGEN
******  DECLARE VARIABLES, PARAMETERS, AND INTRINSIC FUNCTIONS
      DOUBLE PRECISION CHART ( 1:1000 ), SAMPLE ( 1:1000 )
      DOUBLE PRECISION L1, L2, L3, L4, P
      CHARACTER ANSWER
      INTEGER NUM, I
******  NORMAL DISTRIBUTION PARAMETERS
******  mean=0, sigma 2=1, skewness=0, kurtosis=3
      PARAMETER ( L1=0.0D0, L2=0.19750D0, L3=0.13490D0, L4=0.13490D0 )
******  UNIFORM DISTRIBUTION PARAMETERS
******  mean=0, sigma 2=1, skewness=0, kurtosis=1.8
**      PARAMETER ( L1=0.0D0, L2=0.57740D0, L3=1.0D0, L4=1.0D0 )
```

```
****** EXPONENTIAL DISTRIBUTION PARAMETERS
****** mean=0, sigma 2=1, skewness=2, kurtosis=9
**     PARAMETER (L1=-0.9930D0,L2=-0.001081,L3=-0.0000041, L4=-0.00108)
****** STUDENT'S T DISTRIBUTION PARAMETERS
****** mean=0, sigma 2=1, skewness=0, kurtosis=9
**     PARAMETER (L1=0.0D0,L2=-0.32030D0,L3=-0.13590D0,L4=-0.13590D0)
****** WEIBULL DISTRIBUTION PARAMETERS
****** mean=0, sigma 2=1, skewness=1.4, kurtosis=5.8
**     PARAMETER (L1=-0.8440D0,L2=0.05380D0,L3=0.006530D0,L4=0.05470D0)
****** GAMMA DISTRIBUTION PARAMETERS
****** mean=0, sigma 2=1, skewness=1, kurtosis=4.6
**     PARAMETER (L1=-0.6380D0,L2=0.07410D0,L3=0.01820D0,L4=0.06970D0)
****** BETA DISTRIBUTION PARAMETERS
****** mean=0, sigma 2=1, skewness=1.7, kurtosis=5.2
**     PARAMETER ( L1=0.0D0, L2=0.0D0, L3=0.0D0, L4=0.0D0 )
      INTRINSIC MOD
****** PROMPT USER FOR INPUTS
 300  PRINT *, ' '
      PRINT *, 'DO YOU WANT TO GENERATE A DATA SAMPLE FILE?'
      READ *, ANSWER
      IF ((ANSWER.NE.'Y').AND.(ANSWER.NE.'y')) THEN
          GOTO 200
      ENDIF
      PRINT *, ' '
      PRINT *, 'ENTER THE NUMBER OF ELEMENTS IN THE SAMPLE:'
      READ *, NUM
      PRINT *, ' '
      PRINT *, 'ENTER THE SEED FOR THE RANDOM NUMBER GENERATOR:'
      PRINT *, ' '
      PRINT *, '(AN INTEGER BETWEEN 2 AND 2147483647)'
      READ *, SEED
      PRINT *, ' '
****** GENERATE ARRAY OF RANDOM NUMBERS
      CALL DURAND (SEED, NUM, CHART)
****** GENERATE ARRAY OF GLD RANDOM VARIATES
      DO 100 I = 1, NUM
        P = CHART (I)
        SAMPLE (I) = L1 + ((P**L3)-((1-P)**L4))/L2
 100  CONTINUE
      CALL SAMPFILE (NUM, SAMPLE)
      GOTO 300
 200  RETURN
```

```
      END

***************************************************************************
*  FORTRAN 77 CODE USED IN AFIT/ENS MASTER'S THESIS;
*  "THE USE OF L-MOMENTS TO FIT THE GENERALIZED LAMBDA DISTRIBUTION
*   TO SAMPLE DATA"
*  SUBROUTINE TITLE:  DURAND
*  SUBROUTINE PURPOSE:  PSEUDO-RANDOM NUMBER GENERATOR
*  SUBROUTINE DESCRIPTION:
*  This subroutine uses a multiplicative congruential generator with a
*  base of 2**31-1 and a multiplier of 7**5.
*  AUTHOR(S):
*  J. R. M. HOSKING
*  IBM RESEARCH DIVISION
*  T. J. WATSON RESEARCH CENTER
*  YORKTOWN HEIGHTS
*  NEW YORK  10598, USA
*  DATE:  JULY 1988
*  INPUT PARAMETERS:
*  SEED   *IN/OUT* SEED FOR RANDOM NUMBER GENERATOR. SHOULD BE A WHOLE
*                  NUMBER IN THE RANGE 2D0 TO 2147483647D0.
*  N      * INPUT* NUMBER OF NUMBERS TO BE GENERATED
*  OUTPUT PARAMETERS:
*  X      *OUTPUT* ARRAY OF LENGTH N. ON EXIT, CONTAINS RANDOM NUMBERS.
***************************************************************************
      SUBROUTINE DURAND(SEED,N,X)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DOUBLE PRECISION X(N)
      DATA AMULT,BASE/16807D0,2147483647D0/
      DO 10 I=1,N
          SEED=DMOD(SEED*AMULT,BASE)
          X(I)=SEED/BASE
   10 CONTINUE
      RETURN
      END

***************************************************************************
*  FORTRAN 77 CODE USED IN AFIT/ENS MASTER'S THESIS;
*  "THE USE OF L-MOMENTS TO FIT THE GENERALIZED LAMBDA DISTRIBUTION
*   TO SAMPLE DATA"
*  SUBROUTINE TITLE:SAMPFILE
*  SUBROUTINE PURPOSE:  OUTPUT ARRAY OF RANDOM SAMPLE DATA ELEMENTS TO FILE
```

```
*  SUBROUTINE DESCRIPTION:
*  This subroutine inputs an array of random variates from the GLD random
*  sample generator and loads them into an output file.  The file name is
*  input by the user when prompted by this routine.
*  AUTHOR(S):
*  R. B. MOHAN, CAPTAIN, USAF
*  DEPARTMENT OF OPERATIONAL SCIENCES
*  GRADUATE SCHOOL OF ENGINEERING, AIR FORCE INSTITUTE OF TECHNOLOGY
*  WRIGHT-PATTERSON AIR FORCE BASE
*  OHIO  45433, USA
*  DATE:  OCTOBER 1993
*  INPUT PARAMETERS:
*       NUM:    INTEGER PARAMETER number of points in set
*       LAMBDA: DOUBLE PRECISION PARAMETER data array
*       NAME:   CHARACTER STRING output file name
*  OUTPUT PARAMETERS:  NONE.
******************************************************************************
      SUBROUTINE SAMPFILE ( NUM, LAMBDA )
******  DECLARE LOCAL VARIABLES
      DOUBLE PRECISION LAMBDA ( 1:1000 )
      INTEGER  NUM, I, II
      CHARACTER*12 NAME
******  PROMPT USER FOR NAME OF FILE
      PRINT *, 'ENTER THE NAME OF THE OUTPUT FILE:'
      READ (*, '(A12)' ) NAME
******  OPEN OUTPUT FILE AND DOWNLOAD FORMATTED DATA
      OPEN (UNIT=30, FILE=NAME, STATUS='NEW', IOSTAT=II, ERR=200)
  10  FORMAT (F9.4)
******  download array
      DO 100 I = 1, NUM
        WRITE (30, 10) LAMBDA (I)
 100  CONTINUE
******  close output file
      CLOSE (30)
******  RETURN TO MAIN PROGRAM
      RETURN
******  ERROR TRAP
 200  PRINT *, 'CANNOT OPEN FILE ', NAME, ', ERROR= ', II
      RETURN
      END
```

Appendix L. FORTRAN Programs: Data Sample Input and Sort Routines

```
******************************************************************************
*  FORTRAN 77 CODE USED IN AFIT/ENS MASTER'S THESIS;
*  "THE USE OF L-MOMENTS TO FIT THE GENERALIZED LAMBDA DISTRIBUTION
*   TO SAMPLE DATA"
*  SUBROUTINE TITLE:  POINTSET
*  SUBROUTINE PURPOSE:  Input a sample data file into an array
*  SUBROUTINE DESCRIPTION:
*  This subroutine prompts the user for the name of the input file, opens the
*  data input file, and retreives the set of points that make up the sample
*  data.  It then loads the points into an array.  The array is limited to
*  1000 points.
*  AUTHOR(S):
*  R. B. MOHAN, CAPTAIN, USAF
*  DEPARTMENT OF OPERATIONAL SCIENCES
*  GRADUATE SCHOOL OF ENGINEERING, AIR FORCE INSTITUTE OF TECHNOLOGY
*  WRIGHT-PATTERSON AIR FORCE BASE
*  OHIO  45433, USA
*  DATE:  OCTOBER 1993
*  INPUT PARAMETERS:  NONE.
*  OUTPUT PARAMETERS:
*  CHART - 1000 element array containing points from sample data file.
*  NUM - integer containing number of elements in the CHART array.
******************************************************************************
      SUBROUTINE POINTSET ( CHART, NUM )
******  DECLARE LOCAL VARIABLES
      INTEGER  NUM, I, II
      DOUBLE PRECISION  CHART ( 1:1000 )
      CHARACTER*12 NAME
******  PROMPT USER FOR NAME OF FILE
      PRINT *, ' '
      PRINT *, 'ENTER THE NAME OF THE SAMPLE DATA FILE:'
      READ (*, '(A12)' ) NAME
******  OPEN INPUT FILE AND RETREIVE DATA
      OPEN (UNIT=10, FILE=NAME, STATUS='OLD', IOSTAT=II, ERR=130)
******  READ ELEMENTS INTO ARRAY
      NUM = 0
      DO 110 I = 1, 1000
        READ (10,100, IOSTAT=II, ERR=140, END=120) CHART (I)
 100    FORMAT (F9.4)
        NUM = NUM + 1
```

```
 110  CONTINUE
      PRINT *, '1000 ELEMENTS READ INTO ARRAY'
******  CLOSE INPUT FILE
 120  CLOSE (10)
******  RETURN TO MAIN PROGRAM
      RETURN
******  ERROR TRAP
 130  PRINT *, 'CANNOT OPEN FILE ', NAME, ', ERROR= ', II
      RETURN
 140  PRINT *, 'CANNOT READ FILE ', NAME, ', ERROR= ', II
      RETURN
      END

*****************************************************************************
*  FORTRAN 77 CODE USED IN AFIT/ENS MASTER'S THESIS;
*  "THE USE OF L-MOMENTS TO FIT THE GENERALIZED LAMBDA DISTRIBUTION
*   TO SAMPLE DATA"
*  SUBROUTINE TITLE:  SORT
*  SUBROUTINE PURPOSE:  SORT AN ARRAY INTO ASCENDING ORDER
*  SUBROUTINE DESCRIPTION:
*  This subroutine sorts the input array of real numbers into ascending order
*  and outputs the sorted array.  It does not use a temporary array.  The
*  method used is a shell sort with a sequence of increments.
*  AUTHOR(S):
*  J. R. M. HOSKING
*  IBM RESEARCH DIVISION
*  T. J. WATSON RESEARCH CENTER
*  YORKTOWN HEIGHTS
*  NEW YORK  10598, USA
*  DATE:  JULY 1988
*  INPUT PARAMETERS:
*  X - Array of N elements, double precision real numbers, contains the
*  sample data elements to be sorted.
*  N - Integer containing the number of elements in the data sample array.
*  OUTPUT PARAMETERS:
*  X - Array of N elements, double precision real numbers, contains the
*  sorted sample data elements.
*  N - Integer containing the number of elements in the data sample array.
*****************************************************************************
      SUBROUTINE SORT (X,N)
******  DECLARE VARIABLES
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
```

```
      DOUBLE PRECISION X(N)
****** ERROR CHECKING
      IF(N.LE.1)RETURN
      J=4
      DO 10 I=1,100
          J=3*J+1
          IF(J.GE.N)GOTO 20
   10 CONTINUE
   20 CONTINUE
      M=(J/3)
      DO 60 MM=1,100
          M=M/3
          IF(M.EQ.0)RETURN
          DO 50 I=M+1,N
              TEST=X(I)
              J=I
              DO 30 JJ=1,100
                  J=J-M
                  IF(J.LE.0)GOTO 40
                  IF(TEST.GE.X(J))GOTO 40
   30             X(J+M)=X(J)
   40         CONTINUE
   50     X(J+M)=TEST
   60 CONTINUE
      END
```

Appendix M. FORTRAN Program: Sample L-moment Computer

```
*****************************************************************************
*  FORTRAN 77 CODE USED IN AFIT/ENS MASTER'S THESIS;
*  "THE USE OF L-MOMENTS TO FIT THE GENERALIZED LAMBDA DISTRIBUTION
*   TO SAMPLE DATA"*
*  SUBROUTINE TITLE:  SAMLMR
*  SUBROUTINE PURPOSE:  COMPUTE THE SAMPLE L-MOMENTS OF A SAMPLE DATA ARRAY
*  SUBROUTINE DESCRIPTION:
*  This subroutine receives the number of L-moments desired (up to 20) & array
*  of sample data and computes the L-moments for that sample data.  It begins
*  by computing the order statistics and applying the plotting positions.
*  Then it computes the L-moment functions of the order statistics.
*  FOR UNBIASED ESTIMATES (OF THE LAMBDA'S) SET A=B=ZERO. OTHERWISE,
*  PLOTTING-POSITION ESTIMATORS ARE USED, BASED ON THE PLOTTING POSITION
*  (J+A)/(N+B)  FOR THE J'TH SMALLEST OF N OBSERVATIONS. FOR EXAMPLE,
*  A=-0.35D0 AND B=0.0D0 YIELDS THE ESTIMATORS RECOMMENDED BY
*  HOSKING ET AL. (1985, TECHNOMETRICS) FOR THE GEV DISTRIBUTION.
*  AUTHOR(S):
*  J. R. M. HOSKING
*  IBM RESEARCH DIVISION
*  T. J. WATSON RESEARCH CENTER
*  YORKTOWN HEIGHTS
*  NEW YORK  10598, USA
*  DATE:  JULY 1988
*  INPUT PARAMETERS:
*  X      * INPUT* ARRAY OF LENGTH N. CONTAINS THE DATA, IN ASCENDING ORDER.
*  N      * INPUT* NUMBER OF DATA VALUES
*  NMOM   * INPUT* NUMBER OF L-MOMENTS TO BE FOUND. AT MOST MAX(N,20).
*  A      * INPUT* ) PARAMETERS OF PLOTTING
*  B      * INPUT* ) POSITION (SEE ABOVE)
*  OUTPUT PARAMETERS:
*  XMOM   *OUTPUT* ARRAY OF LENGTH NMOM. ON EXIT, CONTAINS THE SAMPLE
*                  L-MOMENTS L-1, L-2, T-3, T-4, ... .
*****************************************************************************
      SUBROUTINE SAMLMR(X,N,XMOM,NMOM,A,B)
******  DECLARE VARIABLES AND INITIALIZE PARAMETERS
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DOUBLE PRECISION X(N),XMOM(NMOM),SUM(20)
      DATA ZERO/0D0/,ONE/1D0/
******  ERROR CHECKING:  Too many L-moments for routine or sample size
      IF(NMOM.GT.20.OR.NMOM.GT.N)GOTO 1000
```

```
****** INITIALIZE ARRAY VALUES TO ZERO
      DO 10 J=1,NMOM
   10 SUM(J)=ZERO
****** BYPASS PLOTTING-POSITION ROUTINE IF A AND B ARE ZERO
      IF(A.EQ.ZERO.AND.B.EQ.ZERO)GOTO 50
****** ERROR CHECKING:  A <= -1 or a >= B
      IF(A.LE.-ONE.OR.A.GE.B)GOTO 1010
*         PLOTTING-POSITION ESTIMATES OF PWM'S
*
*         PLOTTING-POSITION (PPOS) = Ith Value of N values +
*                                    Position Adjustment A /
*                                    N (sample size) +
*                                    Position Adjustment B
      DO 30 I=1,N
*                                    for each element of the sample,
          PPOS=(I+A)/(N+B)
*                                    compute the plotting position,
          TERM=X(I)
          SUM(1)=SUM(1)+TERM
*                                    compute the sum for the mean (L-1).
          DO 20 J=2,NMOM
*                                    for second L-moment and higher,
              TERM=TERM*PPOS
*                                    the element x plotting position,
   20         SUM(J)=SUM(J)+TERM
*                                    add the term to the running total
*                                    for the Jth L-moment.
   30 CONTINUE
****** DIVIDE EACH SUM VALUE BY NUMBER OF VALUES IN ARRAY (AVERAGE)
      DO 40 J=1,NMOM
   40     SUM(J)=SUM(J)/N
****** SKIP NEXT SECTION
      GOTO 100
****** UNBIASED ESTIMATES OF PWM'S (NOT USED WITH GLD)
   50 DO 70 I=1,N
          Z=I
          TERM=X(I)
          SUM(1)=SUM(1)+TERM
          DO 60 J=2,NMOM
              Z=Z-ONE
              TERM=TERM*Z
   60         SUM(J)=SUM(J)+TERM
```

```
   70 CONTINUE
      Y=N
      Z=N
      SUM(1)=SUM(1)/Z
      DO 80 J=2,NMOM
          Y=Y-ONE
          Z=Z*Y
   80     SUM(J)=SUM(J)/Z
******  L-MOMENTS:  Compute functions of PWM's
  100 K=NMOM
      P0=ONE
      IF(NMOM-NMOM/2*2.EQ.1)P0=-ONE
      DO 120 KK=2,NMOM
*                                        for second L-moment and higher,
          AK=K
          P0=-P0
          P=P0
          TEMP=P*SUM(1)
*                                        load mean with adjusted sign.
          DO 110 I=1,K-1
*                                        for remaining L-moments,
              AI=I
              P=-P*(AK+AI-ONE)*(AK-AI)/(AI*AI)
*                                        compute function coefficients,
  110         TEMP=TEMP+P*SUM(I+1)
*                                        add to running total of linear
*                                        function.
          SUM(K)=TEMP
  120     K=K-1
******  COMPUTE MEAN OF SAMPLE
      XMOM(1)=SUM(1)
******  IF ONLY 1ST MOMENT IS WANTED, RETURN
      IF(NMOM.EQ.1)RETURN
      XMOM(2)=SUM(2)
******  ERROR CHECK FOR NON-VARIABLE SAMPLE (NO VARIANCE)
      IF(SUM(2).EQ.ZERO)GOTO 1020
      IF(NMOM.EQ.2)RETURN
******  COMPUTE TAU VALUES (TAU-K = LAMBDA-K / LAMBDA-2)
      DO 130 K=3,NMOM
  130     XMOM(K)=SUM(K)/SUM(2)
      RETURN
******  ERROR TRAPS
```

```
1000 WRITE(6,7000)
     RETURN
1010 WRITE(6,7010)
     RETURN
1020 WRITE(6,7020)
     RETURN
7000 FORMAT(' *** ERROR *** ROUTINE SAMLMR : PARAMETER NMOM INVALID')
7010 FORMAT(' *** ERROR *** ROUTINE SAMLMR :',
    *  ' PLOTTING-POSITION PARAMETERS INVALID')
7020 FORMAT(' *** ERROR *** ROUTINE SAMLMR : ALL DATA VALUES EQUAL')
     END
```

Appendix N. FORTRAN Program: L-Moment to Lambda Parameter Computer

```
**********************************************************************************
*  FORTRAN 77 CODE USED IN AFIT/ENS MASTER'S THESIS;
*  "THE USE OF L-MOMENTS TO FIT THE GENERALIZED LAMBDA DISTRIBUTION
*   TO SAMPLE DATA"
*  SUBROUTINE TITLE:  PELTU4
*  SUBROUTINE PURPOSE:  PARAMETER ESTIMATION VIA L-MOMENTS FOR THE 4-PARAMETER
*                       GENERALIZED LAMBDA DISTRIBUTION
*  SUBROUTINE DESCRIPTION:
*  This subroutine receives the first four sample L-moments from a data sample
*  and computes the four lambda parameters of the GLD using a Newton-Raphson
*  search routine for solving two equations with two unknowns.
*  IN GENERAL, TWO SETS OF PARAMETERS CAN BE FOUND WHICH YIELD A GIVEN
*  SET OF L-MOMENTS. THE ROUTINE LOOKS ONLY FOR PARAMETERS WHICH SATISFY
*  -1.LT.BETA.LE.+1 AND -1.LT.DELTA.LE.+1. SUCH PARAMETERS DO NOT EXIST
*  FOR ALL SETS OF L-MOMENTS, BUT ARE UNIQUE WHEN THEY DO EXIST.
*  THE BETA AND DELTA PARAMETERS ARE ESTIMATED USING NEWTON-RAPHSON
*  ITERATION ON THE RELATIONSHIP BETWEEN (TAU-3,TAU-4) AND (BETA,DELTA).
*  THE CONVERGENCE CRITERION IS THAT TAU-3 AND TAU-4 CALCULATED FROM THE
*  ESTIMATED VALUES OF BETA AND DELTA SHOULD DIFFER BY LESS THAN 1E-6
*  FROM THE VALUES SUPPLIED IN ARRAY XMOM. UNLESS A PARAMETER IS CLOSE
*  TO +1, THE RETURNED PARAMETER ESTIMATES WILL ALSO BE CORRECT TO
*  WITHIN 1E-6.
*  AUTHOR(S):
*  J. R. M. HOSKING
*  IBM RESEARCH DIVISION
*  T. J. WATSON RESEARCH CENTER
*  YORKTOWN HEIGHTS
*  NEW YORK  10598, USA
*  R. B. MOHAN, CAPTAIN, USAF
*  DEPARTMENT OF OPERATIONAL SCIENCES
*  GRADUATE SCHOOL OF ENGINEERING, AIR FORCE INSTITUTE OF TECHNOLOGY
*  WRIGHT-PATTERSON AIR FORCE BASE
*  OHIO  45433, USA
*  DATE:  OCTOBER 1993
*  INPUT PARAMETERS:
*  XMOM    * INPUT* ARRAY OF LENGTH 4. CONTAINS THE L-MOMENTS IN THE ORDER
*                 LAMBDA-1, LAMBDA-2, TAU-3, TAU-4.
*  OUTPUT PARAMETERS:
```

```
*  PARA   *OUTPUT* ARRAY OF LENGTH 5. CONTAINS THE ESTIMATED PARAMETERS
*                  IN THE ORDER XI, ALPHA, BETA, GAMMA, DELTA.
*                  ALPHA AND GAMMA ARE EQUAL.
*  GLD    *OUTPUT* ARRAY OF LENGTH 4.  CONTAINS THE ESTIMATED PARAMETERS
*                  IN THE ORDER LAMBDA-1, -2, -3, -4.
*                  (THESE ARE RAMBERG'S GLD PARAMETERS)
*  IFAIL  *OUTPUT* FAIL FLAG. ON EXIT, IT IS SET TO:
*                  0 ON SUCCESSFUL EXIT
*                  1 IF L-MOMENTS WERE INVALID
*                  2 IF L-MOMENTS WERE NOT CONSISTENT WITH THE
*                      RESTRICTIONS -1.LT.BETA.LT.1, -1.LT.DELTA.LT.1
*                  3 IF ESTIMATES COULD NOT BE FOUND (ITERATION FAILED
*                        TO CONVERGE)
*******************************************************************************
      SUBROUTINE PELTU4(XMOM, PARA, GLD, IFAIL)
******  DECLARE VARIABLES AND INITIALIZE PARAMETERS
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 XMOM(4),PARA(5), GLD(4)
      DATA EPS/1D-6/,MAXIT/20/
* EPS IS TOLERANCE OF APPROXIMATION
* MAXIT IS MAX. NUMBER OF ITERATIONS
* LOAD LOCAL VARIABLES WITH INPUT
      T3=XMOM(3)
      T4=XMOM(4)
* INITIALIZE OUTPUT ARRAY WITH ZEROES
      DO 10 I=1,5
   10     PARA(I)=0D0
******  TEST FOR FEASIBILITY AND ERRORS
* CHECK FOR VARIANCE LESS THAN OR EQUAL TO ZERO
      IF(XMOM(2).LE.0D0)GOTO 1000
      IF(DABS(T3).GE.1D0.OR.DABS(T4).GE.1D0)GOTO 1000
      IF(T4.LT.1.25*T3*T3-0.25)GOTO 1000
      IF(T4.LT.0D0)GOTO 1010
      X=DABS(T3)
      IF(X*(5*X-T4)*(3*X*X-X*T4+2*T4*T4).GT.(5*T4-X)*(X+T4)**2)GOTO 1010
*         SET STARTING VALUES FOR N-R ITERATION
*         (ALGORITHM CONVERGES FAST, SO EVEN A CRUDE GUESS IS OK)
* INITIAL APPROXIMATIONS OF BETA AND DELTA
      B=0D0
      D=0D0
******  NEWTON-RAPHSON ITERATION
      DO 100 IT=1,MAXIT
```

```
****** CALCULATE CURRENT ESTIMATES OF TAU-3 AND TAU-4
      B1=1D0/(1D0+B)
      B2=1D0/(2D0+B)
      B3=1D0/(3D0+B)
      B4=1D0/(4D0+B)
      D1=1D0/(1D0+D)
      D2=1D0/(2D0+D)
      D3=1D0/(3D0+D)
      D4=1D0/(4D0+D)
      X2=B1*B2
      X3=X2*(1D0-B)*B3
      X4=X3*(2D0-B)*B4
      Y2=D1*D2
      Y3=Y2*(1D0-D)*D3
      Y4=Y3*(2D0-D)*D4
      AL2= X2+Y2
      AL3=-X3+Y3
      AL4= X4+Y4
      TAU3=AL3/AL2
      TAU4=AL4/AL2
****** TEST FOR CONVERGENCE
* DIFFERENCE BETWEEN INPUT AND ESTIMATED TAU-3 & TAU-4 VALUES
      E1=TAU3-T3
      E2=TAU4-T4
      IF(DABS(E1).LT.EPS.AND.DABS(E2).LT.EPS)GOTO 120
*         - NOT CONVERGED: CALCULATE NEXT STEP
*         --- NOTATION:
*             DL2B - DERIVATIVE OF LAMBDA-2 W.R.T. B
*             DT3B - DERIVATIVE OF TAU-3 W.R.T. B
*             G..  - MATRIX OF DERIVATIVES
*             H..  - INVERSE OF G
*             DEL. - STEPLENGTH
      BB1=1D0/(1D0-B)
      BB2=1D0/(2D0-B)
      DD1=1D0/(1D0-D)
      DD2=1D0/(2D0-D)
      DL2B=-X2*(B1+B2)
      DL3B= X3*(B1+B2+B3+BB1)
      DL4B=-X4*(B1+B2+B3+B4+BB1+BB2)
      DL2D=-Y2*(D1+D2)
      DL3D=-Y3*(D1+D2+D3+DD1)
      DL4D=-Y4*(D1+D2+D3+D4+DD1+DD2)
```

```
      DT3B=(DL3B-TAU3*DL2B)/AL2
      DT3D=(DL3D-TAU3*DL2D)/AL2
      DT4B=(DL4B-TAU4*DL2B)/AL2
      DT4D=(DL4D-TAU4*DL2D)/AL2
* MATRIX G
      G11=DT3B
      G12=DT3D
      G21=DT4B
      G22=DT4D
* DETERMINANT OF G
      DET=G11*G22-G12*G21
* MATRIX H ( INVERSE OF MATRIX G )
      H11= G22/DET
      H12=-G12/DET
      H21=-G21/DET
      H22= G11/DET
* PRODUCT OF VECTOR E AND MATRIX H
      DEL1=E1*H11+E2*H12
      DEL2=E1*H21+E2*H22
*         - TAKE THE STEP
      B=B-DEL1
      D=D-DEL2
*         - IF GONE TOO FAR, REDUCE STEPLENGTH
*     WITH UP TO TEN HALF-STEPS BACKWARDS
      IF(DABS(B).LE.1D0.AND.DABS(D).LE.1D0)GOTO 100
      DO 50 I= 1,10
          DEL1=0.5*DEL1
          DEL2=0.5*DEL2
          B=B+DEL1
          D=D+DEL2
          IF(DABS(B).LE.1D0.AND.DABS(D).LE.1D0)GOTO 100
   50 CONTINUE
*         END OF N-R ITERATION
  100 CONTINUE
*         NOT CONVERGED
      IFAIL=3
      RETURN
*         CONVERGED
  120 IFAIL=0
      PARA(5)=D
      PARA(4)=XMOM(2)/((B*X2)+(D*Y2))
      PARA(3)=B
```

```
      PARA(2)=PARA(4)
      PARA(1)=XMOM(1)+PARA(2)*(D1-B1)
******  COMPUTE BERGEVIN'S GLD PARAMETERS FROM HOSKING'S PARAMETERS
      GLD(1)=PARA(1)
      GLD(2)=1.0D0/PARA(2)
      GLD(3)=PARA(3)
      GLD(4)=PARA(5)
      RETURN
******  ERROR TRAPS
 1000 IFAIL=1
      RETURN
 1010 IFAIL=2
      RETURN
      END
```

Appendix O. FORTRAN Programs: L-Moment to GLD Parameter Computer

```
*****************************************************************************
*  FORTRAN 77 CODE USED IN AFIT/ENS MASTER'S THESIS;
*  "THE USE OF L-MOMENTS TO FIT THE GENERALIZED LAMBDA DISTRIBUTION
*   TO SAMPLE DATA"
*  SUBROUTINE TITLE:  POWELL
*  SUBROUTINE PURPOSE:  COMPUTE FOUR GLD PARAMETERS FROM FIRST FOUR L-MOMENTS
*                       USING POWELL'S ALGORITHM
*  SUBROUTINE DESCRIPTION:
*  This subroutine searches for the four parameters of the
*  Generalized Lambda Distribution corresponding to
*  specified values for the resulting l-moments
*  of the distribution.
*  AUTHOR(S):
*  E. F. MYKYTKA
*  DEPARTMENT OF OPERATIONAL SCIENCES
*  GRADUATE SCHOOL OF ENGINEERING, AIR FORCE INSTITUTE OF TECHNOLOGY
*  WRIGHT-PATTERSON AIR FORCE BASE
*  OHIO  45433, USA
*  DATE:  MARCH 1993
*  INPUT PARAMETERS:
*  LMOM - DOUBLE PRECISION ARRAY of four L-moments
*  OUTPUT PARAMETERS:
*  LAMBDA - DOUBLE PRECISION ARRAY of four GLD parameters
*****************************************************************************
      SUBROUTINE POWELL ( LMOM, LAMBDA )
C  DECLARE VARIABLES
      IMPLICIT REAL*8 (A-H, O-Z)
      REAL*8 LAMBDA(4), MEAN, LMOM(4)
      DIMENSION X(2), E(2), W(10), START(2)
      COMMON ALPHA3, A3, ALPHA4, A4, MEAN, VAR
C        E -- Vector ofvalues used to determine
C             convergence of Powell's algorithm
C   ESCALE -- Step-size multiplier
C    MAXIT -- Maximum number of iterations for
C             Powell's algorithm
C    START -- Vector of starting values for the lambda
C             parameters used by search routine
C  INITIALIZE TOLERANCE VALUE
```

```
      TOL = 2.D-4
C  MAXIMUM NUMBER OF ITERATIONS
      MAXIT = 500
C  STEP-SIZE MULTIPLIER
      ESCALE = 1000
C  CONVERGENCE RESOLUTION VALUES
      E(1) = 0.000010D0
      E(2) = 0.000010D0
C  LOAD LOCAL VARIABLES WITH INPUT PARAMETERS
      MEAN = LMOM(1)
      VAR = LMOM(2)
      A3 = LMOM(3)
      A4 = LMOM(4)
C  STARTING POINTS FOR SEARCH ROUTINE
      START(1) = 0.050D0
      START(2) = 0.050D0
C  INITIALIZATION FOR POWELL'S ALGORITHM
      X(1) = START(1)
      X(2) = START(2)
      NI = 5
      NO = 6
C  CALL SUBROUTINE BOTM TO PERFORM POWELL'S ALGORITHM
      CALL BOTM(X,E,2,EF,ESCALE,3,MAXIT,W,NI,NO,10)
C  INSURE THAT ALPHA3 AND ALPHA4 CORRESPOND
C  TO OPTIMAL VALUES FOR LAMBDA(3) AND LAMBDA(4)
C  RETURNED FROM SUBROUTINE BOTM
      CALL CALCFX(2, X, EF)
      LAMBDA(3) = X(1)
      LAMBDA(4) = X(2)
C  CALCULATE LAMBDA(1) AND LAMBDA(2) FROM LAMBDA(3)
C  AND LAMBDA(4)
      CALL FN2(LAMBDA)
      RETURN
      END

****************************************************************************
*  FORTRAN 77 CODE USED IN AFIT/ENS MASTER'S THESIS;
*  "THE USE OF L-MOMENTS TO FIT THE GENERALIZED LAMBDA DISTRIBUTION
*   TO SAMPLE DATA"
*  SUBROUTINE TITLE:  CALCFX
*  SUBROUTINE PURPOSE:  CALCULATE THE OBJECTIVE FUNCTION FOR POWELL'S
*                       ALGORITHM
```

```
*  SUBROUTINE DESCRIPTION:
*  Evaluates the objective function to be minimized,
*  which is the sum of the squared differences between
*  the desired and calculated values of skewness and
*  kurtosis
*  AUTHOR(S):
*  E. F. MYKYTKA
*  DEPARTMENT OF OPERATIONAL SCIENCES
*  GRADUATE SCHOOL OF ENGINEERING, AIR FORCE INSTITUTE OF TECHNOLOGY
*  WRIGHT-PATTERSON AIR FORCE BASE
*  OHIO  45433, USA
*  DATE:  MARCH 1993
*  INPUT PARAMETERS:
*  L - DOUBLE PRECISION ARRAY two variables of function
*  OUTPUT PARAMETERS:
*  FN - DOUBLE PRECISION function value
*  N - INTEGER count of subroutine calls
*******************************************************************************
      SUBROUTINE CALCFX ( N, L, FN )
C  DECLARE VARIABLES
      IMPLICIT REAL*8 (A-H, O-Z)
      REAL*8 L(2), LAMBDA(4), MEAN
      COMMON ALPHA3, A3, ALPHA4, A4, MEAN, VAR
C  INSURE THAT LAMBDA(3) AND LAMBDA(4) HAVE SAME SIGN
      LAMBDA(3) = L(1)
      LAMBDA(4) = L(2)
      IF (LAMBDA(3)*LAMBDA(4).LT.0D0) GOTO 999
C COMPUTE OBJECTIVE FUNCTION
      B=L(1)
      D=L(2)
      B1=1D0/(1D0+B)
      B2=1D0/(2D0+B)
      B3=1D0/(3D0+B)
      B4=1D0/(4D0+B)
      D1=1D0/(1D0+D)
      D2=1D0/(2D0+D)
      D3=1D0/(3D0+D)
      D4=1D0/(4D0+D)
      X2=B*B1*B2
      X3=X2*(1D0-B)*B3
      X4=X3*(2D0-B)*B4
      Y2=D*D1*D2
```

```
      Y3=Y2*(1D0-D)*D3
      Y4=Y3*(2D0-D)*D4
      AL2= X2+Y2
      AL3=-X3+Y3
      AL4= X4+Y4
      ALPHA3=AL3/AL2
      ALPHA4=AL4/AL2
C  DIFFERENCE BETWEEN DESIRED AND CALCULATED VALUES
      DIFF3 = ALPHA3 - A3
      DIFF4 = ALPHA4 - A4
C  FUNCTION VALUE IS SQUARED DIFFERENCE BETWEEN
C  CALCULATED AND DESIRED ALPHA3 AND ALPHA4
      FN = DIFF3*DIFF3 + DIFF4*DIFF4
      RETURN
 999  FN = 10.D0
      RETURN
      END

******************************************************************************
*  FORTRAN 77 CODE USED IN AFIT/ENS MASTER'S THESIS;
*  "THE USE OF L-MOMENTS TO FIT THE GENERALIZED LAMBDA DISTRIBUTION
*   TO SAMPLE DATA"
*  SUBROUTINE TITLE:FN2
*  SUBROUTINE PURPOSE:  CALCULATE LAMBDA-1 AND LAMBDA-2  AS FUNCTIONS OF
*                       LAMBDA-3 AND LAMBDA-4
*  SUBROUTINE DESCRIPTION:
*  This subroutine receives an incomplete array of Lambda Parameters and
*  completes the array as a function of the parameters provided.
*  AUTHOR(S):
*  R. B. MOHAN, CAPTAIN, USAF
*  DEPARTMENT OF OPERATIONAL SCIENCES
*  GRADUATE SCHOOL OF ENGINEERING, AIR FORCE INSTITUTE OF TECHNOLOGY
*  WRIGHT-PATTERSON AIR FORCE BASE
*  OHIO  45433, USA
*  E. F. MYKYTKA
*  DEPARTMENT OF OPERATIONAL SCIENCES
*  GRADUATE SCHOOL OF ENGINEERING, AIR FORCE INSTITUTE OF TECHNOLOGY
*  WRIGHT-PATTERSON AIR FORCE BASE
*  OHIO  45433, USA
*  DATE:  OCTOBER 1993
*  INPUT PARAMETERS:
*  LAMBDA - DOUBLE PRECISION ARRAY of four Lambda Parameters
```

```
*  OUTPUT PARAMETERS:
*  LAMBDA - DOUBLE PRECISION ARRAY of four Lambda Parameters
*********************************************************************************
      SUBROUTINE FN2 ( LAMBDA )
C  DECLARE VARIABLES
      IMPLICIT REAL*8 (A-H, O-Z)
      REAL*8 MEAN, LAMBDA(4)
      COMMON ALPHA3, A3, ALPHA4, A4, MEAN, VAR
C  COMPUTE LAMBDA-1 AND LAMBDA-2
      B = LAMBDA(3)
      D = LAMBDA(4)
      B1=1D0/(1D0+B)
      B2=1D0/(2D0+B)
      D1=1D0/(1D0+D)
      D2=1D0/(2D0+D)
      X2=B1*B2
      Y2=D1*D2
C  LOAD OUTPUT PARAMETERS AND RETURN
      LAMBDA(2)=VAR/((B*X2)+(D*Y2))
      LAMBDA(1)=MEAN+LAMBDA(2)*(D1-B1)
      LAMBDA(2)=1.0DO/LAMBDA(2)
      RETURN
      END
```

Appendix P. FORTRAN Program: Powell's Algorithm for function minimization

```
**************************************************************************
*  FORTRAN 77 CODE USED IN AFIT/ENS MASTER'S THESIS;
*  "THE USE OF L-MOMENTS TO FIT THE GENERALIZED LAMBDA DISTRIBUTION
*   TO SAMPLE DATA"
*  SUBROUTINE TITLE:  BOTM
*  SUBROUTINE PURPOSE:  MINIMIZE OBJECTIVE FUNCTION USING POWELL'S ALGORITHM
*  SUBROUTINE DESCRIPTION:
* Subroutine performs Powell's Algorithm for function
* minimization, developed by M. J. D. Powell
*   Source:  Keuster, J. L. and Mize, J. H.,
*            OPTIMIZATION TECHNIQUES WITH FORTRAN,
*            New York: McGraw-Hill, 1973
*  AUTHOR(S):
*  E. F. MYKYTKA
*  DEPARTMENT OF OPERATIONAL SCIENCES
*  GRADUATE SCHOOL OF ENGINEERING, AIR FORCE INSTITUTE OF TECHNOLOGY
*  WRIGHT-PATTERSON AIR FORCE BASE
*  OHIO  45433, USA
*  DATE:  MARCH 1993
*  INPUT/OUTPUT PARAMETERS:
*    X - DOUBLE PRECISION ARRAY OF [illegible], function variables
*    E - DOUBLE PRECISION ARRAY OF S[illegible]2, convergence variables
*    N - INTEGER VARIABLE, 2
*    EF - DOUBLE PRECISION VARIABLE
*    ESCALE - DOUBLE PRECISION VARIABLE, step-size multiplier
*    IPRINT - INTEGER VARIABLE
*    MAXIT - INTEGER VARIABLE, maximum number of iterations
*    W - DOUBLE PRECISION ARRAY OF SIZE NW
*    NI - INTEGER VARIABLE, 5
*    NO - INTEGER VARIABLE, 6
*    NW - INTEGER VARIABLE, 10
**************************************************************************
      SUBROUTINE BOTM (X, E, N, EF, ESCALE, IPRINT,
     +MAXIT, W, NI, NO, NW)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION X(N), W(NW), E(N)
      DDMAG=0.1D0*ESCALE
      SCER=0.05D0/ESCALE
```

```
      JJ=N*(N+1)
      JJJ=JJ+N
      K=N+1
      NFCC=1
      IND=1
      INN=1
      DO 4 I=1,N
      W(I)=ESCALE
      DO 4 J=1,N
      W(K)=0.D0
      IF(I-J)4,3,4
    3 W(K)=DABS (E(I))
    4 K=K+1
      ITERC=1
      ISGRAD=2
          CALL CALCFX(N,X,F)
      FKEEP=2.D0*DABS (F)
    5 ITONE=1
      FP=F
      SUM=0.D0
      IXP=JJ
      DO 6 I=1,N
      IXP=IXP+1
    6 W(IXP)=X(I)
      IDIRN=N+1
      ILINE=1
    7 DMAX=W(ILINE)
      DACC=DMAX*SCER
      DMAG=DMIN1 (DDMAG,0.1D0*DMAX)
      DMAG=DMAX1(DMAG,20.D0*DACC)
      DDMAX=10.D0*DMAG
      GO TO (70,70,71),ITONE
   70 DL=0.D0
      D=DMAG
      FPREV=F
      IS=5
      FA=FPREV
      DA=DL
    8 DD=D-DL
      DL=D
   58 K=IDIRN
      DO 9 I=1,N
```

```
   X(I)=X(I)+DD*W(K)
 9 K=K+1
       CALL CALCFX(N,X,F)
   NFCC=NFCC+1
   GO TO (10,11,12,13,14,96),IS
14 IF(F-FA)15,16,24
16 IF (DABS (D)-DMAX) 17,17,18
17 D=D+D
   GO TO 8
18 WRITE(*,019)
19 FORMAT (5X,38HMAXIMUM CHANGE DOES NOT ALTER FUNCTION)
   GO TO 20
15 FB=F
   DB=D
   GO TO 21
24 FB=FA
   DB=DA
   FA=F
   DA=D
21 GO TO (83,23),ISGRAD
23 D=DB+DB-DA
   IS=1
   GO TO 8
83 D=0.5D0*(DA+DB-(FA-FB)/(DA-DB))
   IS=4
   IF((DA-D)*(D-DB))25,8,8
25 IS=1
   IF(DABS (D-DB)-DDMAX)8,8,26
26 D=DB+DSIGN (DDMAX,DB-DA)
   IS=1
   DDMAX=DDMAX+DDMAX
   DDMAG=DDMAG+DDMAG
   IF (DDMAG.GE.1.D+60)  DDMAG = 1.D+60
   IF(DDMAX-DMAX)8,8,27
27 DDMAX=DMAX
   GO TO 8
13 IF(F-FA)28,23,23
28 FC=FB
   DC=DB
29 FB=F
   DB=D
   GO TO 30
```

```
12 IF(F-FB)28,28,31
31 FA=F
   DA=D
   GO TO 30
11 IF(F-FB)32,10,10
32 FA=FB
   DA=DB
   GO TO 29
71 DL=1.D0
   DDMAX=5.
   FA=FP
   DA=-1.D0
   FB=FHOLD
   DB=0.D0
   D=1.D0
10 FC=F
   DC=D
30 A=(DB-DC)*(FA-FC)
   B=(DC-DA)*(FB-FC)
   IF((A+B)*(DA-DC))33,33,34
33 FA=FB
   DA=DB
   FB=FC
   DB=DC
   GO TO 26
34 D=0.5D0*(A*(DB+DC)+B*(DA+DC))/(A+B)
   DI=DB
   FI=FB
   IF(FB-FC)44,44,43
43 DI=DC
   FI=FC
44 GO TO (86,86,85),ITONE
85 ITONE=2
   GO TO 45
86 IF (DABS (D-DI)-DACC) 41,41,93
93 IF (DABS (D-DI)-0.03D0*DABS (D)) 41,41,45
45 IF ((DA-DC)*(DC-D)) 47,46,46
46 FA=FB
   DA=DB
   FB=FC
   DB=DC
   GO TO 25
```

```
 47 IS=2
    IF ((DB-D)*(D-DC)) 48,8,8
 48 IS=3
    GO TO 8
 41 F=FI
    D=DI-DL
    DD=DSQRT ((DC-DB)*(DC-DA)*(DA-DB)/(A+B))
    DO 49 I=1,N
    X(I)=X(I)+D*W(IDIRN)
    W(IDIRN)=DD*W(IDIRN)
 49 IDIRN=IDIRN+1
    W(ILINE)=W(ILINE)/DD
    ILINE=ILINE+1
    IF(IPRINT-1)51,50,51
 50 CONTINUE
    GO TO (51,53,53),IPRINT
 51 GO TO (55,38),ITONE
 55 IF (FPREV-F-SUM) 94,95,95
 95 SUM=FPREV-F
    JIL=ILINE
 94 IF (IDIRN-JJ) 7,7,84
 84 GO TO (92,72),IND
 92 FHOLD=F
    IS=6
    IXP=JJ
    DO 59 I=1,N
    IXP=IXP+1
 59 W(IXP)=X(I)-W(IXP)
    DD=1.DO
    GO TO 58
 96 GO TO (112,87),IND
112 IF(FP-F) 37,37,91
 91 D=2.DO*(FP+F-2.DO*FHOLD)/(FP-F)**2
    IF (D*(FP-FHOLD-SUM)**2-SUM) 87,37,37
 87 J=JIL*N+1
    IF (J-JJ) 60,60,61
 60 DO 62 I=J,JJ
    K=I-N
 62 W(K)=W(I)
    DO 97 I=JIL,N
 97 W(I-1)=W(I)
 61 IDIRN=IDIRN-N
```

```
      ITONE=3
      K=IDIRN
      IXP=JJ
      AAA=0.D0
      DO 67 I=1,N
      IXP=IXP+1
      W(K)=W(IXP)
      IF (AAA-DABS (W(K)/E(I))) 66,67,67
   66 AAA=DABS (W(K)/E(I))
   67 K=K+1
      DDMAG=1.D0
      W(N)=ESCALE/AAA
      ILINE=N
      GO TO 7
   37 IXP=JJ
      AAA=0.D0
      F=FHOLD
      DO 99 I=1,N
      IXP=IXP+1
      X(I)=X(I)-W(IXP)
      IF(AAA*DABS (E(I))-DABS (W(IXP))) 98,99,99
   98 AAA=DABS (W(IXP)/E(I))
   99 CONTINUE
      GO TO 72
   38 AAA=AAA*(1.D0+DI)
      GO TO (72,106),IND
   72 IF(IPRINT-2)53,50,50
   53 GO TO (109,88),IND
  109 IF (AAA - 0.1D0) 20,20,76
   76 IF(F-FP)35,78,78
   78 WRITE(*,80)
   80 FORMAT(5X,31HACCURACY LIMITED BY ERRORS IN F)
      GO TO 20
   88 IND=1
   35 DDMAG=0.4D0*DSQRT(DABS(FP-F))
      IF (DDMAG.GE.1.D+60)  DDMAG = 1.D+60
      ISGRAD=1
  108 ITERC=ITERC+1
      IF(ITERC-MAXIT)5,5,81
   81 WRITE(*,82) MAXIT
   82 FORMAT(I5,29H ITERATIONS COMPLETED BY BOTM)
      IF(F-FKEEP)20,20,110
```

```
110 F=FKEEP
    DO 111 I=1,N
    JJJ=JJJ+1
111 X(I)=W(JJJ)
    GO TO 20
101 JIL=1
    FP=FKEEP
    IF(F-FKEEP)105,78,104
104 JIL=2
    FP=F
    F=FKEEP
105 IXP=JJ
    DO 113 I=1,N
    IXP=IXP+1
    K=IXP+N
    GO TO (114,115),JIL
114 W(IXP)=W(K)
    GO TO 113
115 W(IXP)=X(I)
    X(I)=W(K)
113 CONTINUE
    JIL=2
    GO TO 92
106 IF(AAA-0.1D0) 20,20,107
 20 EF=F
    RETURN
107 INN=1
    GO TO 35
    END
```

Appendix Q. FORTRAN Programs: Report File Generator

```
******************************************************************************
*  FORTRAN 77 CODE USED IN AFIT/ENS MASTER'S THESIS;
*  "THE USE OF L-MOMENTS TO FIT THE GENERALIZED LAMBDA DISTRIBUTION
*   TO SAMPLE DATA"
*  SUBROUTINE TITLE:  REPORT
*  SUBROUTINE PURPOSE:  GENERATE REPORT OUTPUT FILE
*  SUBROUTINE DESCRIPTION:
*  This subroutine collects all the information generated by the program
*  and outputs it to a text file in a readable and labeled format.  The
*  subroutine prompts the user for a filename.  Output includes:
*    number of elements in sample
*    First four L-moments of sample
*    Hosking's Lambda Parameters from Newton-Raphson Search
*    Ramberg's Lambda Parameters derived from Hosking's Parameters
*    Ramberg's Lambda Parameters from Powell's Algorithm
*    Re-computed L-moments from N-R search
*    Re-computed L-moments from Powell's algorithm
*  AUTHOR(S):
*  R. B. MOHAN, CAPTAIN, USAF
*  DEPARTMENT OF OPERATIONAL SCIENCES
*  GRADUATE SCHOOL OF ENGINEERING, AIR FORCE INSTITUTE OF TECHNOLOGY
*  WRIGHT-PATTERSON AIR FORCE BASE
*  OHIO  45433, USA
*  DATE:  OCTOBER 1993
*  INPUT PARAMETERS:
*       NUM:    INTEGER PARAMETER number of points in set
*       I, II:  INTEGER PARAMETER failure codes
*       MOMENT: DOUBLE PRECISION PARAMETER data array - L-moments
*       LAMBDA: DOUBLE PRECISION PARAMETER data array - Hosking's parameters
*       NEWA:   DOUBLE PRECISION PARAMETER data array - N-R L-moments
*       NEWB:   DOUBLE PRECISION PARAMETER data array - Powell L-moments
*       GLD:    DOUBLE PRECISION PARAMETER data array - Ramberg's parameters
*       POWGLD: DOUBLE PRECISION PARAMETER data array - Ramberg's parameters
*  OUTPUT PARAMETERS:  NONE.
*
******************************************************************************
      SUBROUTINE REPORT (NUM,MOMENT,LAMBDA,GLD,POWGLD,NEWA,NEWB,I)
******  DECLARE LOCAL VARIABLES
      DOUBLE PRECISION MOMENT (4), LAMBDA (5), NEWA (4), GLD (4)
      DOUBLE PRECISION POWGLD (4), NEWB (4)
```

```
      INTEGER  NUM, I, II
      CHARACTER*12 NAME
******  PROMPT USER FOR NAME OF FILE
      PRINT *, 'ENTER THE NAME OF THE OUTPUT FILE:'
      READ (*, '(A12)' ) NAME
******  OPEN OUTPUT FILE AND DOWNLOAD FORMATTED DATA
      OPEN (UNIT=30, FILE=NAME, STATUS='NEW', IOSTAT=II, ERR=200)
******  format
   10 FORMAT (10X)
   20 FORMAT ('NUMBER OF ELEMENTS IN SAMPLE DATA: ', I4)
   30 FORMAT ('FIRST FOUR MOMENTS OF SAMPLE DATA')
   40 FORMAT ('L-1: ', G13.6)
   50 FORMAT ('L-2: ', G13.6)
   60 FORMAT ('T-3: ', G13.6)
   70 FORMAT ('T-4: ', G13.6)
   80 FORMAT ('LAMBDA DISTRIBUTION PARAMETERS (Hosking)')
   90 FORMAT ('XI:    ', G13.6)
  100 FORMAT ('ALPHA: ', G13.6)
  110 FORMAT ('BETA:  ', G13.6)
  120 FORMAT ('GAMMA: ', G13.6)
  130 FORMAT ('DELTA: ', G13.6)
  140 FORMAT ('L-MOMENTS ARE INVALID')
  150 FORMAT ('L-MOMENTS ARE INCONSISTENT')
  160 FORMAT ('FAILURE TO CONVERGE')
  170 FORMAT ('RE-COMPUTED L-MOMENTS (Newton-Raphson)')
  180 FORMAT ('GLD DISTRIBUTION PARAMETERS (Bergevin)')
  210 FORMAT ('Lambda-1: ', G13.6)
  220 FORMAT ('Lambda-2: ', G13.6)
  230 FORMAT ('Lambda-3: ', G13.6)
  240 FORMAT ('Lambda-4: ', G13.6)
  250 FORMAT ('GLD DISTRIBUTION PARAMETERS (Powell)')
  260 FORMAT ('RE-COMPUTED L-MOMENTS (Powell)')
****** write to output file
      WRITE (30,10)
      WRITE (30,10)
      WRITE (30,20) NUM
      WRITE (30,10)
      WRITE (30,30)
      WRITE (30,10)
      WRITE (30,40) MOMENT(1)
      WRITE (30,10)
      WRITE (30,50) MOMENT(2)
```

```
WRITE (30,10)
WRITE (30,60) MOMENT(3)
WRITE (30,10)
WRITE (30,70) MOMENT(4)
WRITE (30,10)
IF (I .GT. 0) GOTO 190
WRITE (30,80)
WRITE (30,10)
WRITE (30,90) LAMBDA(1)
WRITE (30,10)
WRITE (30,100) LAMBDA(2)
WRITE (30,10)
WRITE (30,110) LAMBDA(3)
WRITE (30,10)
WRITE (30,120) LAMBDA(4)
WRITE (30,10)
WRITE (30,130) LAMBDA(5)
WRITE (30,10)
WRITE (30,180)
WRITE (30,10)
WRITE (30,210) GLD(1)
WRITE (30,10)
WRITE (30,220) GLD(2)
WRITE (30,10)
WRITE (30,230) GLD(3)
WRITE (30,10)
WRITE (30,240) GLD(4)
WRITE (30,10)
WRITE (30,250)
WRITE (30,10)
WRITE (30,210) POWGLD(1)
WRITE (30,10)
WRITE (30,220) POWGLD(2)
WRITE (30,10)
WRITE (30,230) POWGLD(3)
WRITE (30,10)
WRITE (30,240) POWGLD(4)
WRITE (30,10)
WRITE (30,170)
WRITE (30,10)
WRITE (30,40) NEWA(1)
WRITE (30,10)
```

```
      WRITE (30,50) NEWA(2)
      WRITE (30,10)
      WRITE (30,60) NEWA(3)
      WRITE (30,10)
      WRITE (30,70) NEWA(4)
      WRITE (30,10)
      WRITE (30,260)
      WRITE (30,10)
      WRITE (30,40) NEWB(1)
      WRITE (30,10)
      WRITE (30,50) NEWB(2)
      WRITE (30,10)
      WRITE (30,60) NEWB(3)
      WRITE (30,10)
      WRITE (30,70) NEWB(4)
      WRITE (30,10)
      WRITE (30,10)
 190  IF (I.EQ.1) WRITE (30,140)
      IF (I.EQ.2) WRITE (30,150)
      IF (I.EQ.3) WRITE (30,160)
******  close output file
      CLOSE (30)
******  RETURN TO MAIN PROGRAM
      RETURN
******  ERROR TRAP
 200  PRINT *, 'CANNOT OPEN FILE ', NAME, ', ERROR= ', II
      RETURN
      END

*****************************************************************************
*  FORTRAN 77 CODE USED IN AFIT/ENS MASTER'S THESIS;
*  "THE USE OF L-MOMENTS TO FIT THE GENERALIZED LAMBDA DISTRIBUTION
*   TO SAMPLE DATA"
*  SUBROUTINE TITLE:  RECOMP
*  SUBROUTINE PURPOSE:  RE-COMPUTES L-MOMENTS FROM LAMBDA PARAMETERS TO
*                       DOUBLE-CHECK ACCURACY OF SEARCH ROUTINES
*  SUBROUTINE DESCRIPTION:
*  This subroutine receives the four lambda parameters from a search routine
*  and runs them back through the defining equations for the L-moments to
*  recompute the L-moments.  This acts as a check of the accuracy of the
*  search routines.
*  AUTHOR(S):
```

```
*  R. B. MOHAN, CAPTAIN, USAF
*  DEPARTMENT OF OPERATIONAL SCIENCES
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*  WRIGHT-PATTERSON AIR FORCE BASE
*  OHIO  45433, USA
*  DATE:  OCTOBER 1993
*  INPUT PARAMETERS:
*  LAMBDA - Double Precision Array of four lambda parameters
*  OUTPUT PARAMETERS:
*  NEWMOM - Double Precision Array of four L-moments
*********************************************************************************
      SUBROUTINE RECOMP ( LAMBDA, NEWMOM )
******  DECLARE LOCAL VARIABLES
      DOUBLE PRECISION LAMBDA ( 1:5 ), NEWMOM ( 1:4 )
      DOUBLE PRECISION L1, L2, L3, L4, T3, T4, X, A, B, C, D, P1, P2
******  INITIALIZE LOCAL VARIABLES
      X = LAMBDA (1)
      A = LAMBDA (2)
      B = LAMBDA (3)
      C = LAMBDA (4)
      D = LAMBDA (5)
******  CALCULATE L-MOMENTS FROM LAMBDA PARAMETERS
      P1 = (A / (1+B))
      P2 = (C / (1+D))
      L1 = X + P1 - P2
      P1 = P1 * (B / (2+B))
      P2 = P2 * (D / (2+D))
      L2 = P1 + P2
      P1 = P1 * ((B-1) / (3+B))
      P2 = P2 * ((D-1) / (3+D))
      T3 = ( P1 - P2 ) / L2
      P1 = P1 * ((B-2) / (4+B))
      P2 = P2 * ((D-2) / (4+D))
      T4 = ( P1 + P2 ) / L2
******  LOAD OUTPUT PARAMETERS
      NEWMOM (1) = L1
      NEWMOM (2) = L2
      NEWMOM (3) = T3
      NEWMOM (4) = T4
******  RETURN TO MAIN PROGRAM
      RETURN
      END
```

Appendix R. FORTRAN Program: PDF Plot File Generator

```
******************************************************************************
*  FORTRAN 77 CODE USED IN AFIT/ENS MASTER'S THESIS;
*  "THE USE OF L-MOMENTS TO FIT THE GENERALIZED LAMBDA DISTRIBUTION
*   TO SAMPLE DATA"
*  SUBROUTINE TITLE:  PLOTFILE
*  SUBROUTINE PURPOSE:  GENERATE DATA OUTPUT FILE TO FEED MATLAB PLOTTING
*                       ROUTINE TO GENERATE GRAPH
*  SUBROUTINE DESCRIPTION:
*  This subroutine receives the four GLD parameters and generates a two-
*  dimensional array of plotting pairs.  There are 999 pairs of p (0 to 1)
*  vs. R(p), determined by the GLD percentile function and the input GLD
*  parameters.  The subroutine then prompts the user for a file name and
*  outputs the array to the file.
*  AUTHOR(S):
*  R. B. MOHAN, CAPTAIN, USAF
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*  GRADUATE SCHOOL OF ENGINEERING, AIR FORCE INSTITUTE OF TECHNOLOGY
*  WRIGHT-PATTERSON AIR FORCE BASE
*  OHIO  45433, USA
*  DATE:  OCTOBER 1993
*  INPUT PARAMETERS:
*  GLD - DOUBLE PRECISION ARRAY of four Lambda Parameters
*  OUTPUT PARAMETERS:  NONE.
******************************************************************************
      SUBROUTINE PLOTFILE ( GLD )
******  DECLARE LOCAL VARIABLES
      DOUBLE PRECISION GLD ( 1:4 )
      DOUBLE PRECISION PLOT (1:999, 1:2 )
      DOUBLE PRECISION L1, L2, L3, L4, P, X, FX
      INTEGER  NUM, I, II
      CHARACTER ANSWER
      CHARACTER*12 NAME
******  INITIALIZE LOCAL VARIABLES
      P = ODO
      NUM = 999
      L1 = GLD (1)
      L2 = GLD (2)
      L3 = GLD (3)
      L4 = GLD (4)
******  COMPUTE ARRAY OF PLOTTING POINT PAIRS
```

```
      DO 300 I = 1, NUM
      P = P + 0.0010D0
      X = L1 + ((P**L3) - ((1-P)**L4))/L2
      FX = L2/((L3*(P**(L3-1)))+(L4*((1-P)**(L4-1))))
      PLOT ( I, 1 ) = X
      PLOT ( I, 2 ) = FX
 300  CONTINUE
******  PROMPT USER FOR NAME OF FILE
      PRINT *, 'DO YOU WANT TO GENERATE A DATA PLOTTING FILE?'
      READ *, ANSWER
      IF ((ANSWER.NE.'Y').AND.(ANSWER.NE.'y')) THEN
          GOTO 400
      ENDIF
      PRINT *, 'ENTER THE NAME OF THE PLOT FILE:'
      READ (*, '(A12)' ) NAME
******  OPEN OUTPUT FILE AND DOWNLOAD FORMATTED DATA
      OPEN (UNIT=30, FILE=NAME, STATUS='NEW', IOSTAT=II, ERR=200)
  10  FORMAT (F11.5, 2X, F11.5)
******  download array
      DO 100 I = 1, NUM
        WRITE (30, 10) PLOT (I,1), PLOT (I,2)
 100  CONTINUE
******  close output file
      CLOSE (30)
******  RETURN TO MAIN PROGRAM
 400  RETURN
******  ERROR TRAP
 200  PRINT *, 'CANNOT OPEN FILE ', NAME, ', ERROR= ', II
      RETURN
      END
```

Appendix S. FORTRAN Programs: CDF Plot File Generator

```
******************************************************************************
*  FORTRAN 77 CODE USED IN AFIT/ENS MASTER'S THESIS;
*  "THE USE OF L-MOMENTS TO FIT THE GENERALIZED LAMBDA DISTRIBUTION
*   TO SAMPLE DATA"
*  SUBROUTINE TITLE:CDFPLOT
*  SUBROUTINE PURPOSE:  BUILDS PLOTFILE FOR CDF OF GLD FROM LAMBDA PARAMETERS
*  SUBROUTINE DESCRIPTION:
*  This subroutine builds an array of plotting points of the CDF of a
*  theoretical distribution using internal lambda parameters and an array of
*  plotting points of the CDF of an estimated distribution using external
*  lambda parameters.  It then downloads the array to an output file.  It
*  also determines Kolmogorov-Smirnov Statistics by comparing the two
*  GLD CDF's: MIN, MAX, and AVG values of differences between theoretical and
*  empirical CDF's.
*  AUTHOR(S):
*  R. B. MOHAN, CAPTAIN, USAF
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*  WRIGHT-PATTERSON AIR FORCE BASE
*  OHIO  45433, USA
*  DATE:  NOVEMBER 1993
*  INPUT PARAMETERS:
*   LAMBDA: DOUBLE PRECISION PARAMETER data array
*  OUTPUT PARAMETERS: NONE.
******************************************************************************
      SUBROUTINE CDFPLOT ( POWGLD )
******  DECLARE LOCAL VARIABLES
      DOUBLE PRECISION POWGLD (4), THEOGLD (4)
      DOUBLE PRECISION PLOT (1:599, 1:3 )
      DOUBLE PRECISION X, FX1, FX2, DELTA, MIN, MAX, AVG
      INTEGER  NUM, I, II
      CHARACTER ANSWER
      CHARACTER*12 NAME
      DOUBLE PRECISION CDF
      EXTERNAL CDF
******  INITIALIZE LOCAL VARIABLES (normal distribution)
      X = -3.0D0 *  first plotting pair
      NUM = 599 *  number of plotting pairs
      THEOGLD (1) = 0.0D0 *  theoretical mean (lambda 1)
      THEOGLD (2) = 0.19750D0 *  theoretical lambda 2
```

```
      THEOGLD (3) = 0.13490D0 *  theoretical lambda 3
      THEOGLD (4) = 0.13490D0   *  theoretical lambda 4
      MIN = 1.0D0
      MAX = 0.0D0
      AVG = 0.0D0
******  COMPUTE ARRAY OF PLOTTING POINT PAIRS
      DO 300 I = 1, NUM
      X = X + 0.010D0 *  increment between plotting points
      FX1 = CDF ( THEOGLD, X ) *  theoretical cdf value
      FX2 = CDF ( POWGLD, X ) *  empirical cdf value
      PLOT ( I, 1 ) = X
      PLOT ( I, 2 ) = FX1
      PLOT ( I, 3 ) = FX2
******  COMPUTE KOLMOGOROV-SMIRNOV STATISICS
*  compute the difference between the F(x) values for theoretical and
*  empirical distributions
      DELTA = DABS ( FX1 - FX2 )
*  search for the minimum difference between the distributions
      IF ( DELTA .LT. MIN ) MIN = DELTA
*  search for the maximum difference between the distributions
      IF ( DELTA .GT. MAX ) MAX = DELTA
*  compute the average of all the differences between the distributions
      AVG = AVG + DELTA
 300  CONTINUE
      AVG = AVG/NUM
*  output K-S stats to screen
      PRINT *, ' '
      PRINT *, 'K-S Statistics'
      PRINT *, ' '
      PRINT *, 'MINIMUM DELTA-CDF: ', MIN
      PRINT *, 'AVERAGE DELTA-CDF: ', AVG
      PRINT *, 'MAXIMUM DELTA-CDF: ', MAX
      PRINT *, ' '
******  PROMPT USER FOR NAME OF FILE
      PRINT *, 'DO YOU WANT TO GENERATE A CDF DATA PLOTTING FILE?'
      READ *, ANSWER
      IF ((ANSWER.NE.'Y').AND.(ANSWER.NE.'y')) THEN
          GOTO 400
      ENDIF
      PRINT *, 'ENTER THE NAME OF THE CDF PLOT FILE:'
      READ (*, '(A12)' ) NAME
******  OPEN OUTPUT FILE AND DOWNLOAD FORMATTED DATA
```

```
      OPEN (UNIT=30, FILE=NAME, STATUS='NEW', IOSTAT=II, ERR=200)
   10 FORMAT (F11.5, 2X, F11.5, 2X, F11.5)
****** download array
      DO 100 I = 1, NUM
        WRITE (30, 10) PLOT (I,1), PLOT (I,2), PLOT (I,3)
  100 CONTINUE
****** close output file
      CLOSE (30)
****** RETURN TO MAIN PROGRAM
  400 RETURN
****** ERROR TRAP
  200 PRINT *, 'CANNOT OPEN FILE ', NAME, ', ERROR= ', II
      RETURN
      END

*****************************************************************************
*  FORTRAN 77 CODE USED IN AFIT/ENS MASTER'S THESIS;
*  "THE USE OF L-MOMENTS TO FIT THE GENERALIZED LAMBDA DISTRIBUTION
*   TO SAMPLE DATA"
*  SUBROUTINE TITLE:CDF
*  SUBROUTINE PURPOSE:  COMPUTE F(X) VALUES OF CDF OF GLD, GIVEN X VALUES
*  SUBROUTINE DESCRIPTION:
*  This function uses a secant search to find the value of p between 0 and
*  1, given the value of X. It then finds the value of F(X) using the derived
*  value of p.
*  AUTHOR(S):
*  R. B. MOHAN, CAPTAIN, USAF
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*  WRIGHT-PATTERSON AIR FORCE BASE
*  OHIO  45433, USA
*  DATE:  NOVEMBER 1993
*  INPUT PARAMETERS:
*   LAMBDA: DOUBLE PRECISION PARAMETER data array
*        X:  DOUBLE PRECISION PARAMETER independent variable
*  OUTPUT PARAMETERS:
* CDF:  DOUBLE PRECISION PARAMETER dependent variable
```

```
**********************************************************************************
      FUNCTION CDF ( LAMBDA, X )
******  DECLARE LOCAL VARIABLES
      IMPLICIT DOUBLE PRECISION ( A-H, O-Z )
      DOUBLE PRECISION LAMBDA(4)
      DOUBLE PRECISION GLDQF
      EXTERNAL GLDQF
* Check to see if the range is bounded or not
      IF(LAMBDA(3).LT.0.0) GOTO 20
      IF(LAMBDA(4).LT.0.0) GOTO 20
* Range is bounded: find endpoints and, if X is below
* lower endpoint set F(x) = 0; if X is above upper
* endpoint set F(x) = 1
      XLOWER = LAMBDA(1) - 1.0/LAMBDA(2)
      IF(X.GT.XLOWER) GOTO 10
      CDF = 0.0
      RETURN
 10   XUPPER = LAMBDA(1) + 1.0/LAMBDA(2)
      IF(X.LT.XUPPER) GOTO 20
      CDF = 1.0
      RETURN
* Find the value of the CDF at X using the secant
* method to find p such that the inverse CDF evaluated
* at p is X
 20   P1 = 0.4
      P2 = 0.6
      F1 = GLDQF(LAMBDA,P1) - X
      F2 = GLDQF(LAMBDA,P2) - X
      DO 30 N = 1, 200
         PNEW = P2 - (P2 - P1)*F2/(F2 - F1)
         IF(PNEW.LE.0.0) PNEW = .000001
         IF(PNEW.GE.1.0) PNEW = .999999
         IF(PNEW.EQ.P2) GOTO 40
         FNEW = GLDQF(LAMBDA,PNEW) - X
         IF(DABS(FNEW).LE.0.000005) GOTO 40
         P1 = P2
         P2 = PNEW
         F1 = F2
 30      F2 = FNEW
* Secant method does not converge
      WRITE(6,999)
 999  FORMAT(35H0ERROR (CDF): SECANT METHOD DOES NOT,
```

```
     +30H CONVERGE AFTER 200 ITERATIONS)
      RETURN
* Secant method converges to a solution
 40   CDF = PNEW
      RETURN
      END
* GLD Quantile Function
      FUNCTION GLDQF ( LAMBDA, P )
* Declare variables
      IMPLICIT DOUBLE PRECISION ( A-H, O-Z )
      DOUBLE PRECISION LAMBDA(4)
* Error checking
      IF ( P .LT. 0.0D0 ) GOTO 998
      IF ( P .GT. 1.0D0 ) GOTO 998
* Compute GLD quantile function
      TERM = P**LAMBDA(3) - (1.0 - P)**LAMBDA(4)
      GLDQF = LAMBDA(1) + TERM/LAMBDA(2)
      RETURN
* Error trap
  998 WRITE(6,999) P
  999 FORMAT(20H0ERROR (GLDQF): P = , E9.4,
     +32H AND IS NOT BETWEEN ZERO AND ONE)
      RETURN
      END
```

Appendix T. FORTRAN Programs: Conventional Moment Calculator

```
*************************************************************************
*  FORTRAN 77 CODE USED IN AFIT/ENS MASTER'S THESIS;
*  "THE USE OF L-MOMENTS TO FIT THE GENERALIZED LAMBDA DISTRIBUTION
*   TO SAMPLE DATA"
*  SUBROUTINE TITLE:  CALCFX
*  SUBROUTINE PURPOSE:  CALCULATE THE OBJECTIVE FUNCTION FOR POWELL'S
*                       ALGORITHM
*  SUBROUTINE DESCRIPTION:
*  Evaluates the objective function to be minimized,
*  which is the sum of the squared differences between
*  the desired and calculated values of skewness and
*  kurtosis
*  AUTHOR(S):
*  E. F. MYKYTKA
*  DEPARTMENT OF OPERATIONAL SCIENCES
*  GRADUATE SCHOOL OF ENGINEERING, AIR FORCE INSTITUTE OF TECHNOLOGY
*  WRIGHT-PATTERSON AIR FORCE BASE
*  OHIO  45433, USA
*  DATE:  1979
*  INPUT PARAMETERS:
*  L - DOUBLE PRECISION ARRAY two variables of function
*  OUTPUT PARAMETERS:
*  FN - DOUBLE PRECISION function value
*  N - INTEGER count of subroutine calls
*************************************************************************
      SUBROUTINE CALCFX(N, L, FN)
*  DECLARE LOCAL VARIABLES AND GLOBAL PARAMETERS
      IMPLICIT REAL*8 (A-H, O-Z)
      REAL*8 L(2), LAMBDA(4), MEAN
      COMMON ALPHA3, A3, ALPHA4, A4, MEAN, VAR, IFUN
*  COUNT NUMBER TIMES SUBROUTINE IS CALLED
      IFUN = IFUN + 1
*  ENSURE THAT LAMBDA(3) AND LAMBDA(4) HAVE SAME SIGN
      LAMBDA(3) = L(1)
      LAMBDA(4) = L(2)
      IF (LAMBDA(3)*LAMBDA(4).LT.0D0) GOTO 999
* COMPUTE OBJECTIVE FUNCTION
   10 B40 = 1.D0/(4.D0*LAMBDA(3)+1.D0)
      B31 = BETA(3.D0*LAMBDA(3)+1.D0, LAMBDA(4)+1.D0)
      B22 = BETA(2.D0*LAMBDA(3)+1.D0,
```

```
     +               2.DO*LAMBDA(4)+1.DO)
      B13 = BETA(LAMBDA(3)+1.DO, 3.DO*LAMBDA(4)+1.DO)
      B04 = 1.DO/(4.DO*LAMBDA(4)+1.DO)
      D = B40 - 4.DO*B31 + 6.DO*B22 - 4.DO*B13 + B04
   20 B30 = 1.DO/(3.DO*LAMBDA(3)+1.DO)
      B21 = BETA(2.DO*LAMBDA(3)+1.DO, LAMBDA(4)+1.DO)
      B12 = BETA(LAMBDA(3)+1.DO, 2.DO*LAMBDA(4)+1.DO)
      B03 = 1.DO/(3.DO*LAMBDA(4)+1.DO)
      C = B30 - 3.DO*B21 + 3.DO*B12 - B03
   30 B20 = 1.DO/(2.DO*LAMBDA(3)+1.DO)
      B11 = BETA(LAMBDA(3)+1.DO,LAMBDA(4)+1.DO)
      B02 = 1.DO/(2.DO*LAMBDA(4)+1.DO)
      B = B20 - 2.DO*B11 + B02
      B10 = 1.DO/(LAMBDA(3)+1.DO)
      B01 = 1.DO/(LAMBDA(4)+1.DO)
      A = B10 - B01
   40 ALPHA3 = (C - 3*B*A + 2*A**3)/(B - A*A)**1.5
      ALPHA3 = ALPHA3*DSIGN(1.DO, LAMBDA(3))
      IF(LAMBDA(3).NE.0.DO) GOTO 50
      ALPHA3 = ALPHA3*DSIGN(1.DO,LAMBDA(4))
   50 ALPHA4 = D - 4.DO*C*A + 6.DO*B*A*A - 3.DO*A**4
      ALPHA4 = ALPHA4/(B - A*A)**2
      DIFF3 = ALPHA3 - A3
      DIFF4 = ALPHA4 - A4
*  FUNCTION VALUE IS SQUARED DIFFERENCE BETWEEN
*  CALCULATED AND DESIRED ALPHA3 AND ALPHA4
      FN = DIFF3*DIFF3 + DIFF4*DIFF4
      RETURN
  999 FN = 10.DO
      RETURN
      END

******************************************************************************
*  FORTRAN 77 CODE USED IN AFIT/ENS MASTER'S THESIS;
*  "THE USE OF L-MOMENTS TO FIT THE GENERALIZED LAMBDA DISTRIBUTION
*   TO SAMPLE DATA"
*  SUBROUTINE TITLE:  CMOMENTS
*  SUBROUTINE PURPOSE:  CALCULATE THE FIRST FOUR CONVENTIONAL MOMENTS OF A
*                       DATA SAMPLE
*  SUBROUTINE DESCRIPTION:
*  This subroutine receives an array containing the data sample, and computes
*  the mean, variance, skewness, and kurtosis of that sample.  It then
```

```
*  returns those values in an output parameter array.
*  AUTHOR(S):
*  R. B. MOHAN, CAPTAIN, USAF
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*  WRIGHT-PATTERSON AIR FORCE BASE
*  OHIO  45433, USA
*  DATE:  NOVEMBER 1993
*  INPUT PARAMETERS:
*  ARRAY - DOUBLE PRECISION ARRAY sample data
*  LENGTH - INTEGER length of input array (number of elements in sample)
*  OUTPUT PARAMETERS:
*  MOMENT - DOUBLE PRECISION ARRAY first four conventional moments
*****************************************************************************
      SUBROUTINE CMOMENTS ( ARRAY, LENGTH, MOMENT )
*  DECLARE VARIABLES
      INTEGER LENGTH, I
      DOUBLE PRECISION ARRAY ( 1:3000 )
      DOUBLE PRECISION MOMENT ( 1:4 )
      DOUBLE PRECISION MEAN, VAR, SKEW, KURT
*  INITIALIZE VARIABLES
      MEAN = 0
      VAR = 0
      SKEW = 0
      KURT = 0
*  COMPUTE MEAN
      DO 101 I = 1, LENGTH
        MEAN = MEAN + ARRAY (I)
 101  CONTINUE
      MEAN = MEAN/LENGTH
*  COMPUTE VARIANCE, SKEWNESS, AND KURTOSIS SUMS
      DO 102 I = 1, LENGTH
        VAR = VAR + (ARRAY(I)-MEAN)**2
        SKEW = SKEW + (ARRAY(I)-MEAN)**3
        KURT = KURT + (ARRAY(I)-MEAN)**4
 102  CONTINUE
*  DIVIDE SUMS BY NUMBER OF ELEMENTS IN SAMPLE
      VAR = VAR/LENGTH
      SKEW = SKEW/LENGTH
      KURT = KURT/LENGTH
*  COMPUTE THIRD AND FOURTH MOMENTS
      SKEW = SKEW/(VAR**1.5)
```

```
      KURT = KURT/(VAR**2)
*   LOAD OUTPUT PARAMETER ARRAY
      MOMENT(1) =  MEAN
      MOMENT(2) =  VAR
      MOMENT(3) =  SKEW
      MOMENT(4) =  KURT
*   RETURN TO PROGRAM
      RETURN
      END
```

Appendix U. FORTRAN Programs: Alternate Moment Calculator

```
********************************************************************************
*  FORTRAN 77 CODE USED IN AFIT/ENS MASTER'S THESIS;
*  "THE USE OF L-MOMENTS TO FIT THE GENERALIZED LAMBDA DISTRIBUTION
*   TO SAMPLE DATA"
*  SUBROUTINE TITLE:  CALCFX
*  SUBROUTINE PURPOSE:  CALCULATE THE OBJECTIVE FUNCTION FOR POWELL'S
*                       ALGORITHM
*  SUBROUTINE DESCRIPTION:
*  Evaluates the objective function to be minimized,
*  which is the sum of the squared differences between
*  the desired and calculated values of skewness and
*  kurtosis
*  AUTHOR(S):
*  E. F. MYKYTKA
*  DEPARTMENT OF OPERATIONAL SCIENCES
*  GRADUATE SCHOOL OF ENGINEERING, AIR FORCE INSTITUTE OF TECHNOLOGY
*  WRIGHT-PATTERSON AIR FORCE BASE
*  OHIO  45433, USA
*  DATE:  1979
*  INPUT PARAMETERS:
*  L - DOUBLE PRECISION ARRAY two variables of function
*  OUTPUT PARAMETERS:
*  FN - DOUBLE PRECISION function value
*  N - INTEGER count of subroutine calls
********************************************************************************
      SUBROUTINE CALCFX(N, L, FN)
*  DECLARE LOCAL VARIABLES AND GLOBAL PARAMETERS
      IMPLICIT REAL*8 (A-H, O-Z)
      REAL*8 L(2), LAMBDA(4), MEAN
      COMMON ALPHA3, A3, ALPHA4, A4, MEAN, VAR, IFUN
*  COUNT NUMBER TIMES SUBROUTINE IS CALLED
      IFUN = IFUN + 1
*  ENSURE THAT LAMBDA(3) AND LAMBDA(4) HAVE SAME SIGN
      L3 = L(1)+1.0D0
      L4 = L(2)+1.0D0
      IF ((L(1)*L(2)) .LT. 0.0D0) GOTO 999
*  COMPUTE OBJECTIVE FUNCTION
     U05=20.D0*(((1.D0-(0.95D0**L3))/L3)-((0.05D0**L4)/L4))
     U50=2.D0* (((1.D0-(0.5D0**L3 ))/L3)-((0.5D0**L4 )/L4))
     L05=20.D0*(((0.05D0**L3)/L3)+(((0.95D0**L4)-1.D0)/L4))
```

```
      L50=2.D0* (((0.5D0**L3 )/L3)+(((0.5D0**L4 )-1.D0)/L4))
      M50=((0.75D0**L3)-(0.25D0**L3))/L3
      M50=2.D0*(M50+(((0.25D0**L4)-(0.75D0**L4))/L4))
      ALPHA3=(U05-M50)/(M50-L05)
      ALPHA4=(U05-L05)/(U50-L50)
      DIFF3 = ALPHA3 - A3
      DIFF4 = ALPHA4 - A4
*  FUNCTION VALUE IS SQUARED DIFFERENCE BETWEEN
*  CALCULATED AND DESIRED ALPHA3 AND ALPHA4
      FN = DIFF3*DIFF3 + DIFF4*DIFF4
      RETURN
 999  FN = 10.D0
      RETURN
      END

*****************************************************************************
*  FORTRAN 77 CODE USED IN AFIT/ENS MASTER'S THESIS;
*  "THE USE OF L-MOMENTS TO FIT THE GENERALIZED LAMBDA DISTRIBUTION
*   TO SAMPLE DATA"
*  SUBROUTINE TITLE:  AMOMENTS
*  SUBROUTINE PURPOSE:  CALCULATE THE FIRST FOUR ALTERNATE MOMENTS OF A
*                       DATA SAMPLE
*  SUBROUTINE DESCRIPTION:
*  This subroutine receives an array containing the sorted data sample, and
*  computes the mean, variance, Q3, and Q4 of that sample.  It then
*  returns those values in an output parameter array.
*  AUTHOR(S):
*  R. B. MOHAN, CAPTAIN, USAF
*  DEPARTMENT OF OPERATIONAL SCIENCES
*  GRADUATE SCHOOL OF ENGINEERING, AIR FORCE INSTITUTE OF TECHNOLOGY
*  WRIGHT-PATTERSON AIR FORCE BASE
*  OHIO  45433, USA
*  DATE:  NOVEMBER 1993
*  INPUT PARAMETERS:
*  ARRAY - DOUBLE PRECISION ARRAY sample data
*  LENGTH - INTEGER length of input array (number of elements in sample)
*  OUTPUT PARAMETERS:
*  MOMENT - DOUBLE PRECISION ARRAY first four alternate moments
*****************************************************************************
      SUBROUTINE AMOMENTS ( ARRAY, LENGTH, MOMENT )
*  DECLARE VARIABLES
      INTEGER LENGTH, I, A, B
```

```
      DOUBLE PRECISION ARRAY ( 1:3000 )
      DOUBLE PRECISION MOMENT ( 1:4 )
      DOUBLE PRECISION MEAN, VAR, U, L, M, UH, LH
*  INITIALIZE VARIABLES
      MEAN = 0
      VAR = 0
      A = 1
      B = 12
      U = 0
      L = 0
      M = 0
      UH = 0
      LH = 0
*  CALCULATE MEAN
      DO 101 I = 1, LENGTH
        MEAN = MEAN + ARRAY (I)
 101  CONTINUE
      MEAN = MEAN/LENGTH
*  CALCULATE VARIANCE
      DO 102 I = 1, LENGTH
        VAR = VAR + (ARRAY(I)-MEAN)**2
*  CALCULATE Q3 AND Q4 (SAMPLE SIZE OF 25 ONLY)
 102  CONTINUE
       U = U + ARRAY(25) + (ARRAY(24)/4)
       L = L + ARRAY(1)  + (ARRAY(2)/4)
      DO 104 I = 1, B
        UH = UH + ARRAY(I+13)
        LH = LH + ARRAY(I)
 104  CONTINUE
      UH = UH + (ARRAY(13)/2)
      LH = LH + (ARRAY(13)/2)
      DO 105 I = 8, 18
        M = M + ARRAY(I)
 105  CONTINUE
      M = M + (ARRAY(7)  * 0.75D0)
      M = M + (ARRAY(19) * 0.75D0)
      U=U/1.25
      L=L/1.25
      UH=UH/12.5
      LH=LH/12.5
      M=M/12.5
*  LOAD OUTPUT PARAMETER ARRAY
```

```
      MOMENT(1) =  MEAN
      MOMENT(2) =  VAR/LENGTH
      MOMENT(3) =  (U-M)/(M-L)
      MOMENT(4) =  (U-L)/(UH-LH)
*  RETURN TO PROGRAM
      RETURN
      END
```

Appendix V. FORTRAN Program: Output File Statistic Calculator

```
******************************************************************************
*  FORTRAN 77 CODE USED IN AFIT/ENS MASTER'S THESIS;
*  "THE USE OF L-MOMENTS TO FIT THE GENERALIZED LAMBDA DISTRIBUTION
*   TO SAMPLE DATA"
*  SUBROUTINE TITLE:  STAT
*  SUBROUTINE PURPOSE:
*  SUBROUTINE DESCRIPTION:
*  This subroutine receives the output from 30 reports and finds the minimum
*  and maximum values of each of 11 data types in the reports.  It then
*  calculates the mean and variance of the 30 values for each data type.
*  Then it outputs the statistics in a LaTex tabular environment format.
*  AUTHOR(S):
*  R. B. MOHAN, CAPTAIN, USAF
*  DEPARTMENT OF OPERATIONAL SCIENCES
*  GRADUATE SCHOOL OF ENGINEERING, AIR FORCE INSTITUTE OF TECHNOLOGY
*  WRIGHT-PATTERSON AIR FORCE BASE
*  OHIO  45433, USA
*  DATE:  NOVEMBER 1993
*  INPUT PARAMETERS:
*  User inputs name of input data file
*  OUTPUT PARAMETERS:
*  User inputs name of output statistics file
******************************************************************************
      PROGRAM STAT
******  DECLARE LOCAL VARIABLES
      DOUBLE PRECISION DATA (1:11, 1:30)
* 11 values by 30 samples
      DOUBLE PRECISION STATS (1:11, 1:4)
* 11 values by 4 statistics
      INTEGER I, J
      DOUBLE PRECISION MIN, MAX, AVG, VAR
      CHARACTER*12 INFILE, OUTFILE
******  PROMPT USER FOR NAME OF FILE
      PRINT *, 'ENTER THE NAME OF THE INPUT FILE:'
      READ (*, '(A12)' ) INFILE
******  OPEN INPUT FILE AND UPLOAD FORMATTED DATA
      OPEN (UNIT=20, FILE=INFILE, STATUS='OLD')
  10  FORMAT (F13.6)
******  download input file to array
      DO 100 I = 1, 30
```

```
        DO 120 J = 1, 11
          READ (20, 10, END=110 ) DATA (J,I)
 120    CONTINUE
 100  CONTINUE
******  close output file
 110  CLOSE (20)
******  CALCULATE STATISTICS
      DO 200 I = 1, 11
*  INITIALIZE VARIABLES
      MIN = 10000
      MAX = -10000
      AVG = 0
      VAR = 0
*  FIND MIN, MAX, AND SUM OF VALUES
        DO 300 J = 1, 30
          IF (DATA(I,J) .LT. MIN) MIN = DATA(I,J)
          IF (DATA(I,J) .GT. MAX) MAX = DATA(I,J)
          AVG = AVG + DATA(I,J)
 300    CONTINUE
*  DIVIDE SUM BY NUMBER OF VALUES TO FIND MEAN
        AVG = AVG/30
*  CALCULATE VARIANCE
        DO 400 J = 1, 30
          VAR = VAR + (AVG-DATA(I,J))**2
 400    CONTINUE
*  LOAD OUTPUT PARAMETER ARRAY
      STATS(I,1) = MIN
      STATS(I,2) = MAX
      STATS(I,3) = AVG
      STATS(I,4) = VAR/29
 200  CONTINUE
******  PROMPT USER FOR NAME OF FILE
      PRINT *, 'ENTER THE NAME OF THE OUTPUT FILE:'
      READ (*, '(A12)' ) OUTFILE
******  OPEN OUTPUT FILE AND DOWNLOAD FORMATTED DATA
      OPEN (UNIT=30, FILE=OUTFILE, STATUS='NEW')
******  FORMAT FOR LATEX TABULAR ENVIRONMENT
  40  FORMAT ('            & 1st & ',F11.8,' & ',
     +F11.8,' & ',F11.8,' & ',F11.8,' \\\\ \\cline{3-6}')
  41  FORMAT ('Moments of  & 2nd & ',F11.8,' & ',
     +F11.8,' & ',F11.8,' & ',F11.8,' \\\\ \\cline{3-6}')
  42  FORMAT ('Sample Data & 3rd & ',F11.8,' & ',
```

```
     +F11.8,' & ',F11.8,' & ',F11.8,' \\\\ \\cline{3-6}')
  43  FORMAT ('             & 4th & ',F11.8,' & ',
     +F11.8,' & ',F11.8,' & ',F11.8,' \\\\ \\hline')
  44  FORMAT ('             & \\1  & ',F11.8,' & ',
     +F11.8,' & ',F11.8,' & ',F11.8,' \\\\ \\cline{3-6}')
  45  FORMAT ('Lambda       & \\2  & ',F11.8,' & ',
     +F11.8,' & ',F11.8,' & ',F11.8,' \\\\ \\cline{3-6}')
  46  FORMAT ('Parameters   & \\3  & ',F11.8,' & ',
     +F11.8,' & ',F11.8,' & ',F11.8,' \\\\ \\cline{3-6}')
  47  FORMAT ('             & \\4  & ',F11.8,' & ',
     +F11.8,' & ',F11.8,' & ',F11.8,' \\\\ \\hline')
  48  FORMAT ('Kolmogorov- & MIN & ',F11.8,' & ',
     +F11.8,' & ',F11.8,' & ',F11.8,' \\\\ \\cline{3-6}')
  49  FORMAT ('Smirnov     & AVG & ',F11.8,' & ',
     +F11.8,' & ',F11.8,' & ',F11.8,' \\\\ \\cline{3-6}')
  50  FORMAT ('Statistics  & MAX & ',F11.8,' & ',
     +F11.8,' & ',F11.8,' & ',F11.8,' \\\\ \\hline\\hline')
******  download array to output file
      WRITE (30,40) STATS(1,1), STATS(1,2), STATS(1,3), STATS(1,4)
      WRITE (30,41) STATS(2,1), STATS(2,2), STATS(2,3), STATS(2,4)
      WRITE (30,42) STATS(3,1), STATS(3,2), STATS(3,3), STATS(3,4)
      WRITE (30,43) STATS(4,1), STATS(4,2), STATS(4,3), STATS(4,4)
      WRITE (30,44) STATS(5,1), STATS(5,2), STATS(5,3), STATS(5,4)
      WRITE (30,45) STATS(6,1), STATS(6,2), STATS(6,3), STATS(6,4)
      WRITE (30,46) STATS(7,1), STATS(7,2), STATS(7,3), STATS(7,4)
      WRITE (30,47) STATS(8,1), STATS(8,2), STATS(8,3), STATS(8,4)
      WRITE (30,48) STATS(9,1), STATS(9,2), STATS(9,3), STATS(9,4)
      WRITE (30,49) STATS(10,1), STATS(10,2), STATS(10,3), STATS(10,4)
      WRITE (30,50) STATS(11,1), STATS(11,2), STATS(11,3), STATS(11,4)
******  close output file
      CLOSE (30)
      STOP
      END
```

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Vita

Captain Robert B. Mohan was born on 31 July 1962 in Hawarden, Iowa. He moved to California at age six, and grew up there. He earned the degree of Bachelor of Science in Computer Science from the United States Air Force Academy (USAFA) in May 1984. Immediately after graduation and commissioning as a regular officer in the United States Air Force, He attended Undergraduate Pilot Training-Helicopter (UPT-H) at Fort Rucker, Alabama until May, 1985. He then served three tours as a helicopter pilot, flying Sikorsky HH-3E *Jolly Green Giants* for the USAF Air Rescue Service at Tyndall AFB, Florida; Osan AB, Korea; and Patrick AFB, Florida. In August 1992 he was assigned to the Air Force Institute of Technology (AFIT), where he earned the degree of Master of Science in Operations Research. In March 1994, he was assigned to the 57th Test Group, Fighter Weapons Center at Nellis Air Force Base, Nevada.

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REPORT ...

March 1994 Master's Thesis

THE USE OF L-MOMENTS TO FIT THE GENERALIZED LAMBDA DISTRIBUTION TO SAMPLE DATA

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AFIT/GST/ENS/94M-09

Approved for public release; distribution unlimited

The Generalized Lambda Distribution (GLD) is a four-parameter, continuous probability distribution that is useful for simulation analysis. The strengths of the GLD lie in its abilities to approximate many distributions, represent data when the underlying distribution is unknown, and fit or generate random variates. The method of moments is presently the accepted technique for estimating the parameters of this distribution. However, it is sensitive to extreme observations and subject to large sampling variability as the sample size decreases. L-moments are expectations of certain linear combinations of order statistics. They can be used to estimate parameters and quantiles of probability distributions. Their main advantage over conventional moments is that they suffer less from the effects of sampling variability, and are theoretically more robust to outliers than conventional moments. Estimating the parameters of the GLD by matching its L-moments to those of the sample is known as the method of L-moments. This appears to be an attravtive alternative to the method of moments and is developed in this thesis. A Monte Carlo experiment compared the method of L-moments to the methed of conventional moments and a third method which uses alternate measures of symmetry and tailweight. Experiment results showed that L-moments are better than conventional and alternate moments for fitting distributions to sample data, particularly when the skewness and kurtosis of the sample distribution are large.

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Generalized Lambda Distribution, Linear Moments

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